EXISTENCE THEOREMS FOR NONPROJECTIVE COMPLETE ALGEBRAIC VARIETIES

BY
MASAYOSHI NAGATA

The purpose of the present paper is to prove the following two theorems:

THEOREM 1. *Let L be a function field over a ground field k. Assume that* dim L *is not less than* 2. *Assume furthermore that if* dim $L = 2$, *then k is sufficiently large.*[1] *Then there exists a complete normal abstract variety of L which is not projective.*

THEOREM 2. *If n is a natural number not less than* 3, *then there exists a complete nonsingular variety of dimension n which is not projective*; *more explicitly, there exists a nonsingular complete variety of the rational function field of dimension n, which is defined over the prime field and which is not projective.*

We shall remark that, since Zariski [4] proved that a normal abstract surface can be imbedded in a projective surface (as an open subset) if there exists an affine variety which carries all singular points of the given surface, our results give a complete answer for the imbedding problem in one sense. Therefore it will be an important problem to give some sufficient conditions for a given variety to be projective.[2] It will be also an interesting problem to characterize function fields which have nonsingular complete nonprojective varieties.

1. Two lemmas

LEMMA 1. *Let V and V' be varieties. If V is not projective, then $V \times V'$ is not projective.*

Proof. $V \times V'$ contains a nonprojective subvariety $V \times P'$ ($P' \in V'$), and therefore $V \times V'$ is not projective.

LEMMA 2. *Let V be a normal variety with function field L, and let L' be a finite algebraic extension of L. Let V' be the derived normal variety of V in L'. If V' can be imbedded in a projective variety V'', then V can be imbedded in a projective variety.*

Proof. We may assume that V' is an open subset of V''. Let P be a generic point of V over a ground field k, and let $Z(P)$ be $\sum P'_i$, where P'_i form

Received November 15, 1957.

[1] The meaning of "large" will be explained in the course of the proof.

[2] Cf. Chow [2], Chevalley [1], and Weil [3]. On the other hand, the following problem was offered by Chevalley a few years ago:

Assume that a normal variety V satisfies the following condition: For any finite number of points of V, there exists an affine variety which carries them. Can then V be imbedded in a projective variety?