PROJECTIVE TOPOLOGICAL SPACES

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Suppose we have given a category of topological spaces and continuous maps. Let X, Y, and Z be admissible spaces and ϕ and f admissible maps of X into Z and Y into Z respectively. A natural question is whether or not there exists an admissible map ψ of X into Y such that $\phi = f \circ \psi$. One can hardly expect to answer such a question without explicit knowledge of all the data, but it may happen that, for certain spaces X, the answer is always yes provided f satisfies the minimum condition of mapping Y onto Z. Discrete spaces are examples in the category of all spaces and continuous maps. Following the terminology of homological algebra, we shall call such a space projective. In this paper we will determine the projective spaces in the category of compact spaces and continuous maps and discuss the notion of projective resolution for these spaces.

Throughout the paper the word *space* will mean *Hausdorff space*.

1. The necessary condition

We restrict our attention to those categories of spaces and maps for which

(a) All admissible maps are continuous.

(b) If A is an admissible space and $\{p, q\}$ is a two-element space, then $A \times \{p, q\}$ and the projection map of this space onto A are admissible.

(c) If A is an admissible space and B is a closed subspace of A, then B and the inclusion map of B into A are admissible.

These conditions are not stringent and are satisfied by many of the usual categories.

1.1. DEFINITION. A topological space is said to be extremally disconnected if and only if the closure of every open set is again open.

1.2 THEOREM. *In any category of topological spaces and maps satisfying conditions* (a), (b), *and* (c) *above, a projective space is extremally disconnected.*

Proof. Let X be a projective space in such a category. Let G be any open subset of X; we must prove $\bar{G}$ is open.

In $X \times \{p, q\}$ consider the closed set $Y = ((X - G) \times \{p\}) \cup (\bar{G} \times \{q\})$, and its inclusion map i. Let π be the projection of $X \times \{p, q\}$ onto X. Our hypothesis on the category implies that $\pi \circ i$ is an admissible map of Y onto X and that the identity ϕ is an admissible map of X into X. Since X is projective, there is an admissible map ψ of X into Y such that $\phi = \pi \circ i \circ \psi$.

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