SOME SUFFICIENT CONDITIONS FOR A GROUP TO BE NILPOTENT

In commemoration of G. A. Miller

BY

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1. Let

$$G = G_0 > G_1 > G_2 > \cdots > G_m = 1 \tag{1}$$

be a chain of subgroups of the group G. Following Kaloujnine [1], we define the *stability group* of the chain (1) to be the group A of all automorphisms α of G such that

$$(G_i x)^\alpha = G_i x \tag{2}$$

holds for all $x \in G_{i-1}$ and for each $i = 1, 2, \cdots, m$.

If the subgroups G_i are all normal in G, then it is easy to show that A is nilpotent and of class at most $m - 1$. But without some such assumption of normality, the nature of the group A is not so clear. In [1], however, Kaloujnine proved that A is always at least a soluble group, and the length d of its derived series cannot exceed $m - 1$. He remarks of this result that it is "wahrscheinlich nicht endgültig." In fact, we shall find that A is still nilpotent even in the general case. This is stated in

Theorem 1. *The stability group A of any subgroup-chain* (1) *of length m is nilpotent and of class at most $\frac{1}{2}m(m-1)$.*

It was shown in [3] that a nilpotent group A of derived length d must be of class at least 2^{d-1}. Thus Theorem 1 yields the bound

$$d \leqq [\log_2 m(m-1)] \tag{3}$$

for the derived length of the stability group A. This bound never exceeds $m - 1$ and is smaller than $m - 1$ for $m > 5$. Indeed, it is of a smaller order of magnitude as $m \to \infty$. Hence Kaloujnine's theorem follows from (3).

For the class of A we have the bounds $m - 1$ and $\frac{1}{2}m(m-1)$ which apply in the normal case and the general case, respectively. These bounds first differ when $m = 3$. That the difference is significant we show by constructing a group with a subgroup-chain of length 3 for which the stability group is of class 3. It will also be proved that the subgroup of G generated by all the commutators $x^{-1}x^\alpha$ with $x \in G$ and $\alpha \in A$ is always locally nilpotent. This commutator subgroup is known to be always nilpotent in the normal case: cf. [1], Satz 4. We show by an example that this need not be so in the general case.

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