# SOLUTION OF THE BURNSIDE PROBLEM FOR EXPONENT SIX[1]

**In commemoration of G. A. Miller**

BY

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## 1. Introduction

In 1902 Burnside [1] raised the question as to whether a finitely generated group $G$ of exponent $n$ is necessarily finite. $G$ is said to be of exponent $n$ if $g^n = 1$ for every element $g$ of $G$. For $k$ generators $x_1, \cdots, x_k$ there is a group $B(n, k)$ such that every group of exponent $n$ with $k$ generators is a homomorphic image of $B(n, k)$. Here $B(n, k)$ is easily seen to be $F_k/F_k^n$ where $F_k$ is the free group with $k$ generators, and $F_k^n$ is the fully invariant subgroup of $F_k$ generated by all $n^{\text{th}}$ powers of elements of $F_k$.

It is trivial that the Burnside group $B(2, k)$ is Abelian and of order $2^k$. In his original paper Burnside showed that $B(3, k)$ is finite, but did not find the true order of $B(3, k)$. This value is $3^K$, $K = k + \binom{k}{2} + \binom{k}{3}$ and was obtained by Levi and van der Waerden [5]. Burnside showed that $B(4, 2)$ is of order at most $2^{12}$, and Sanov [6] showed that $B(4, k)$ is finite, but the order of $B(4, k)$ is not known.

In this paper it is shown that $B(6, k)$ is finite. The order of $B(6, k)$ is

$$(1.1) \qquad 2^a 3^{b+\binom{b}{2}+\binom{b}{3}}, \qquad a = 1 + (k-1)\cdot 3^{k+\binom{k}{2}+\binom{k}{3}}, \qquad b = 1 + (k-1)\, 2^k.$$

This follows from a result of Philip Hall and Graham Higman [3]. Their results apply to what is known as the restricted Burnside problem. This is the question as to whether there exists a largest finite group $R(n, k)$ of exponent $n$ generated by $k$ elements. If it can be shown that there is a largest finite group $R(n, k)$, then either $B(n, k)$ is infinite or $B(n, k) = R(n, k)$. They have shown that the existence of a largest finite group for each prime power exponent dividing $n$, and any number of generators, implies the existence of a largest finite solvable group of exponent $n$ and any number of generators. The requirement of solvability is superfluous if $n$ is divisible by only two distinct primes, since any such finite group must be solvable. From their theorems and the result of Levi and van der Waerden they obtained the order above for $R(6, k)$. Graham Higman [4] has solved the restricted Burnside problem for exponent five.

## 2. Theorems on groups of exponent three

THEOREM 2.1. *If a group $G$ is generated by elements $x_1, x_2, \cdots, x_n$, and if any four of the $x$'s generate a group of exponent three, then $G$ is of exponent three.*

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