CLOSE-PACKING AND FROTH

In commemoration of G. A. Miller

BY

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Cannon-balls may aid the truth,
But thought's a weapon stronger;
We'll win our battles by its aid;—
Wait a little longer.
CHARLES MACKAY (1814–1889)
("The Good Time Coming")

1. Algebraic introduction

The abstract groups (2, p, q), defined by

$$R^p = S^q = (RS)^2 = 1,$$

or

$$R^p = S^q = T^2 = RST = 1,$$

or

$$S^q = T^2 = (ST)^p = 1,$$

have been studied intensively ever since Hamilton [11] expressed (2, 3, 5) in the form

$$\iota^2 = \kappa^3 = \lambda^5 = 1, \qquad \lambda = \iota\kappa$$

and wrote, "I am disposed to give the name 'Icosian Calculus' to this system of symbols." Dyck ([8, p. 35]; see also [4, p. 407]) expressed the symmetric and alternating groups

$$\mathfrak{S}_3, \quad \mathfrak{A}_4, \quad \mathfrak{S}_4, \quad \mathfrak{A}_5$$

in the form (2, 3, q) with $q = 2, 3, 4, 5$, respectively. Miller [19, p. 117] remarked that the case when $q = 6$ is entirely different. In fact [20], the group (2, p, q) is finite if and only if

1.1 $$(p - 2)(q - 2) < 4.$$

Thus the finite groups in the family are

the dihedral group (2, 2, q) of order $2q$,
the tetrahedral group (2, 3, 3) of order 12,
the octahedral group (2, 3, 4) of order 24,
the icosahedral group (2, 3, 5) of order 60.

The inequality 1.1 is a necessary and sufficient condition for the finiteness of the number c, which we define to be the period of any one of the elements

$$R^2S^2, \quad R^{-1}S^{-1}RS, \quad TSR, \quad S^{-1}TST.$$

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