A CHARACTERIZATION OF THE ONE-DIMENSIONAL UNIMODULAR PROJECTIVE GROUPS OVER FINITE FIELDS[1]

In commemoration of G. A. Miller

BY

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Let q be a power of a prime number, and denote by $LF(2, q)$ the group of all one-dimensional unimodular projectivities over the field Γ_q with q elements, i.e., of all linear fractional transformations

$$z' = \frac{az + b}{cz + d}$$

of determinant 1 with coefficients a, b, c, d in Γ_q. As is well known, $LF(2, q)$ is simple for $q \geqq 4$. The order of $LF(2, q)$ is $q(q + 1)(q - 1)/2$ for odd q and $q(q + 1)(q - 1)$ for even q.

It is our aim to give a group-theoretical characterization of these groups $LF(2, q)$. We shall prove

THEOREM. *Let $\mathfrak{G}$ be a group of finite order g which satisfies the following conditions:*

(I) *g is even;*

(II) *if $\mathfrak{A}$ and $\mathfrak{B}$ are two cyclic subgroups of $\mathfrak{G}$ of even orders, and if $\mathfrak{A} \cap \mathfrak{B} \neq \{1\}$, then there exists a cyclic subgroup $\mathfrak{Z}$ of $\mathfrak{G}$ which includes both $\mathfrak{A}$ and $\mathfrak{B}$;*

(III) *$\mathfrak{G}$ coincides with its commutator subgroup.*

Then $\mathfrak{G} \cong LF(2, q)$ where $q \geqq 4$ is a prime power.

We begin in I with an elementary discussion of groups which satisfy these conditions. Two cases, A and B, have to be considered. In Case A, the 2-Sylow group $\mathfrak{T}$ of $\mathfrak{G}$ is dihedral, and in Case B, $\mathfrak{T}$ is abelian of type $(2, 2, \cdots, 2)$. These two cases are treated in II and III respectively. The final result is obtained by applying a theorem of Zassenhaus [2] concerning doubly transitive permutation groups. We shall give some extensions of our theorem in a subsequent paper.

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[1] The results of this paper were obtained more or less independently by the three authors. Rather than publish three different papers, we preferred to combine our investigations.

The result for Case B has also been obtained by K. A. Fowler in his thesis, University of Michigan, 1952.

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