# CLOSURE AND DISPERSION OF FINITE GROUPS

**In commemoration of G. A. Miller**

BY

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If $\mathfrak{r}$ is a set of primes, then we term $\mathfrak{r}$-group [$\mathfrak{r}$-element] every finite group [every group element] whose order is divisible by primes in $\mathfrak{r}$ only. A group is termed $\mathfrak{r}$-closed, if its set of $\mathfrak{r}$-elements is a characteristic $\mathfrak{r}$-subgroup; and this is equivalent to requiring that products of $\mathfrak{r}$-elements are again $\mathfrak{r}$-elements. Several well known theorems in finite group theory may be interpreted as criteria for $\mathfrak{r}$-closure; and our principal concern in this investigation will be with such criteria.

If $\mathfrak{r}$ is a set of primes, then we denote by $P\mathfrak{r}$ the complementary set of primes (= set of primes prime to $\mathfrak{r}$); and we say that a group is $P\mathfrak{r}$-homogeneous if its elements induce $P\mathfrak{r}$-automorphisms in its $P\mathfrak{r}$-subgroups. It is easy to see that $\mathfrak{r}$-closed groups are $P\mathfrak{r}$-homogeneous; but there exist $P\mathfrak{r}$-homogeneous groups which are not $\mathfrak{r}$-closed. The clarification of this relation is our main problem. The most comprehensive criterion obtained in this direction is Theorem 5.3: The finite group $G$ is $\mathfrak{r}$-closed if, and only if, it is $P\mathfrak{r}$-homogeneous and $\{R, P\}$ is an $\mathfrak{r}$-$p$-group whenever $R$ is a maximal $\mathfrak{r}$-subgroup of $G$, $P$ a $p$-Sylow subgroup of $G$, and $p$ a prime, not in $\mathfrak{r}$.

On our way we have to focus attention on $Pp$-closure (and dually on $p$-closure); and the analysis of $Pp$-closure is closely related to an investigation of groups with the property that all epimorphic images of subgroups of index prime to $p$ are $p$-normal. The auxiliary results obtained here appear to be of independent interest [§4].

By its very definition dispersion is a concatenation of an involved array of closure requirements. We shall, however, show in §1 that dispersion may be reduced essentially to $p$-closure and $Pp$-closure. Combining this reduction theorem with the closure criteria obtained in §§2 to 5 we obtain a number of interesting dispersion criteria in §6.

## Notations

$o(G)$ = order of group $G$.

$o(g)$ = order of group element $g$.

$G'$ = commutator subgroup of $G$.

$G^{(i)}$ = $i^{\text{th}}$ derivative of group $G$ (inductively defined by $G = G^{(0)}$, $G^{(i+1)} = [G^{(i)}]'$).

$ZG$ = center of $G$.

$Z_i G$ = $i^{\text{th}}$ term in ascending central series of $G$ (inductively defined by $Z_0 G = 1$, $Z_{i+1} G/Z_i G = Z[G/Z_i G]$).

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