ON A THEOREM OF ERDÖS AND SZEKERES

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1. Introduction

Suppose h is a given positive integer greater than 1. Let $M(h)$ be the set of all positive integers n such that $p^h \mid n$ for every prime factor p of n. If x is a positive real number, let $N_h(x)$ be the number of elements of $M(h)$ not exceeding x. Erdös and Szekeres [3] proved that for h fixed

$$N_h(x) = x^{1/h} \prod \left(1 + \sum_{m=h+1}^{2h-1} p^{-m/h}\right) + O(x^{1/(h+1)}).$$

(In this paper an unspecified product is understood to be a product over all the primes p, while an O-relation is understood to be with respect to $x \to \infty$ and is not necessarily uniform in the parameters, such as h, that may be involved.) It is the purpose of this paper to point out that considerably more precise information may be easily obtained from known results in the theory of lattice-point problems. The general idea is the familiar one of expressing the given problem in terms of a "nearby" lattice-point problem whose solution is known, that is, we express the Dirichlet series corresponding to the given problem as the product of a Dirichlet series with a comparatively small abscissa of absolute convergence and the Dirichlet series corresponding to the known lattice-point problem.

More specifically, let $c_n = 1$ if $n \in M(h)$ and $c_n = 0$ if n is a positive integer not in $M(h)$, so that $N_h(x) = \sum_{n \leq x} c_n$. Then, using the Euler product for the Riemann zeta-function, we have

$$(1) \quad \begin{aligned} \sum c_n n^{-s} &= \prod \left(1 + \sum_{m=h}^{\infty} p^{-ms}\right) = \zeta(hs) \prod \left(1 + \sum_{m=h+1}^{2h-1} p^{-ms}\right) \\ &= \zeta(hs)\, \zeta((h+1)s) \prod \left(1 + \sum_{m=h+2}^{2h-1} p^{-ms} - \sum_{m=2h+2}^{3h} p^{-ms}\right). \end{aligned}$$

(Throughout this paper the letters m and n stand for positive integers and an unspecified sum is understood to be a sum over all the positive integers.) Continuing this process we obtain

$$(2) \quad \sum c_n n^{-s} = \zeta(hs)\, \zeta((h+1)s) \cdots \zeta((2h-1)s)\, \zeta^{-1}((2h+2)s) \sum f_n n^{-s},$$

where $\sum f_n n^{-s}$ has abscissa of absolute convergence at most $1/(2h+3)$. (Actually $\sum f_n n^{-s} = 1$ if $h = 2$, while $\sum f_n n^{-s}$ has abscissa of absolute convergence exactly equal to $1/(2h+3)$ if $h > 2$. Repetition of the above procedure shows that $\sum c_n n^{-s}$ has a meromorphic continuation in the half-plane $\operatorname{Re} s > 0$.) Suppose now that we put

$$(3) \quad \begin{aligned} \sum c_n n^{-s} &= \sum a_n n^{-s} \sum b_n n^{-s}, \\ \sum a_n n^{-s} &= \zeta(hs)\, \zeta((h+1)s) \cdots \zeta((h+r)s) \end{aligned}$$

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