ON SOME ARITHMETICAL FUNCTIONS

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In a paper on additive arithmetical functions,[1] P. Erdös incidentally states the following result:

($\mathcal{E}_1$) *Let* $\omega(n)$ *be the number of prime divisors of the positive integer* n, *and let* λ *be any irrational number.*

Then the numbers $\lambda\omega(n)$ *are uniformly distributed modulo* 1.

This means that, for $0 \leqq t \leqq 1$, the number of n's less than or equal to x and such that[2]

$$\lambda\omega(n) - I[\lambda\omega(n)] \leqq t$$

is $tx + o[x]$ as x tends to $+\infty$.

P. Erdös adds that the proof is not easy.

($\mathcal{E}_1$) can actually be deduced from a later result of Erdös, say ($\mathcal{E}_2$), concerning the number of integers $n \leqq x$ for which $\omega(n) = k$.[3]

Also a very short proof can be based on the following formula due to Atle Selberg:[4]

As x tends to $+\infty$,

$$\sum_{n \leqq x} z^{\omega(n)} = F(z)x(\log x)^{z-1} + O[x(\log x)^{\Re z-2}],$$

uniformly for $|z| \leqq R$, where R is any positive number and

$$F(z) = \frac{1}{\Gamma(z)} \prod \left[1 + \frac{z}{p-1}\right]\left[1 - \frac{1}{p}\right]^z.$$

We have only to take $z = \exp[2\pi q\lambda i]$, where q is any positive integer, and use a well known theorem of H. Weyl.[5]

However the proof of ($\mathcal{E}_2$) is not very simple, while the proof of Selberg's formula uses the properties of the Riemann Zeta-function in the critical strip.

In the present paper, we shall first give a simple proof of ($\mathcal{E}_1$) which uses only the nonvanishing of $\zeta(s)$ for $\Re s \geqq 1$. We shall also give some generalizations.

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[1] *On the distribution function of additive functions*, Ann. of Math. (2), vol. 47 (1946), pp. 1–20. See p. 2, lines 4 and 5.

[2] $I[u]$ denotes the greatest integer not exceeding u.

[3] *On the integers having exactly k prime factors*, Ann. of Math. (2), vol. 49 (1948), pp. 53–66.

[4] *Note on a paper by L. G. Sathe*, J. Indian Math. Soc., vol. 18 (1954), pp. 83–87.

[5] *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. vol. 77 (1916), pp. 313–352, Satz 1, p. 315.