PROBABILITY THEORY AND THE FIRST BOUNDARY VALUE PROBLEM

Dedicated to Paul Lévy
on the occasion of his seventieth birthday

BY
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In [4] a rather general approach to the first boundary value problem for a class of functions called regular functions was presented, and the application of probability theory to the solution was indicated. In the present paper, this work is carried further, in several directions.

The place of the relativized problem, introduced by Brelot [1] into the study of harmonic functions on a Green space (see also related work in a different context by Feller [5]) is discussed. Boundary limit properties of extremal and minimal regular functions are obtained. Finally, a new characterization in probability terms of upper and lower first boundary value problem solutions is obtained, which makes possible a rather elegant characterization of the resolutive functions. This characterization implies that, in a large class of applications, including for example the case when the regular functions are the solutions of the heat equation, if the domain of the functions has a compact closure in the defining space, every continuous boundary function is resolutive.

1. Review of [4]

The basis for the theory of regular functions in [4], comprised in hypotheses TM1-4 and RS1-4 of that reference, can be summarized as follows. A locally compact separable Hausdorff space R is given, together with a specified class of open subsets of R, called regular sets. The regular sets have compact closures and form a basis for the topology of R. If D is a regular set, with boundary D', and if $\xi \in D$, there is a certain probability measure $\mu(\xi, D, \cdot)$, defined on the Borel subsets of D'. A function u on R is said to be regular if it is continuous and if it is equal at each point ξ of each regular set D to its average over D' with respect to the measure $\mu(\xi, D, \cdot)$. Corresponding definitions of subregular and superregular functions are made.

An additional hypothesis will be used, but only when mentioned explicitly, in discussing the first boundary value problem for regular functions on an open subset D of R. It is a form of the maximum principle, which we denote by $M(D, D')$, and which states that, if u is subregular on D and bounded from above, its supremum on D is a limiting value of u at some point of the boundary D' of D.

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