ON THE INTRINSIC FORM FOR SECOND ORDER DIFFERENTIAL OPERATORS

Dedicated to Paul Lévy
on the occasion of his seventieth birthday

BY
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1. Introduction

A function f will be said to be *in the local domain of the linear operator* $\mathfrak{A}$ at the point s, (in symbols: $f \in D(\mathfrak{A}, s)$), if f *and* $\mathfrak{A}f$ are defined and *continuous* in some neighborhood of s. Similarly, the domain $D(\mathfrak{A}, I)$ of $\mathfrak{A}$ for the interval I consists of all f such that f and $\mathfrak{A}f$ are continuous in I.

The differential operator

$$\mathfrak{A} = aD_s^2 + bD_s + c, \qquad a > 0, \quad D_s = \frac{d}{ds} \tag{1.1}$$

enjoys the following obvious properties:

(1) *Local character*: If $f(s) = 0$ for all s in a neighborhood of the point s_0, then $f \in D(\mathfrak{A}, s_0)$ and $\mathfrak{A}f(s_0) = 0$.

(2) $\mathfrak{A}$ *is nontrivial*: To each point s of the interval of definition there exists an $f \in D(\mathfrak{A}, s)$ such that $f(a) \neq 0$ and $\mathfrak{A}f(s) \neq 0$.

(3) *Weak minimum property*: Let $f \in D(\mathfrak{A}, s)$ be nonnegative in a neighborhood of s and $f(s) = 0$. Then $\mathfrak{A}f(s) \geqq 0$.

In other words, if the point s is both a zero and a local minimum for f, then $\mathfrak{A}f(s) \geqq 0$. For the *pure* differential operator $\mathfrak{A} = aD_s^2 + bD_s$ (where $a > 0$) the property (3) may be sharpened to

(3′) *Strong minimum property*: If $f \in D(\mathfrak{A}, s)$ has a local minimum at s, then $\mathfrak{A}f(s) \geqq 0$.

Various problems have led the author to derive the general form of linear operators in one dimension having these properties [1]. The class of such operators forms a natural generalization of the classical second order differential operators. It has been shown elsewhere, [1], [2], [3], that their use has considerable advantages. The use of the new canonical form renders the theory more satisfactory and at the same time simpler; it achieves an unexpected unification and is more adapted for many applied problems.

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