INVARIANT WEDDERBURN FACTORS

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The Wedderburn principal theorem for a class of algebras states that if an algebra modulo its radical is separable, then it contains a subalgebra with the same structure as the difference algebra. We wish to investigate the problem of when such a subalgebra is invariant under a group of operators on the algebra. The natural setting for this question is that of the extensions of an algebra. In section 2, conditions are given on the groups and algebras considered, which guarantee the existence of such subalgebras. In section 3, a special case of the main theorem for alternative algebras is used to give a proof of the Wedderburn principal theorem for Jordan algebras of characteristic not two. In section 4, a uniqueness theorem is given for a special case: self-adjoint Wedderburn factors of an associative algebra over a field of characteristic zero.

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1. Preliminaries

Let $\mathfrak{A}$ denote a (finite-dimensional) not necessarily associative algebra over a field Φ. The concept of an extension of $\mathfrak{A}$ is found in [4], and we assume familiarity with the discussion given there. The extension $(\mathfrak{B}, \sigma)$ of $\mathfrak{A}$ with kernel $\mathfrak{K}$ may be represented by the diagram

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{B} \xrightarrow{\sigma} \mathfrak{A} \longrightarrow 0.$$

Recall that σ is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$. If $\mathfrak{K}^2 = \{0\}$, the extension is said to be singular. If $\mathfrak{K}$ is nilpotent, the extension is said to be nilpotent. $\mathfrak{A}$ is segregated in $\mathfrak{B}$ if $\mathfrak{B}$ contains a subalgebra $\mathfrak{A}'$ such that $\mathfrak{B} = \mathfrak{A}' + \mathfrak{K}$, $\mathfrak{A}' \cong \mathfrak{B}/\mathfrak{K}$, and $\mathfrak{A}' \cap \mathfrak{K} = \{0\}$. Such a subalgebra $\mathfrak{A}'$ of $\mathfrak{B}$ will be called a *Wedderburn factor of* $\mathfrak{B}$. $\mathfrak{A}$ is *segregated* if it is segregated in every extension.

We will say $\mathfrak{A}$ is semi-simple if it is the direct sum of simple algebras with nonzero squares, and define the radical of $\mathfrak{A}$ as the minimal ideal $\mathfrak{N}$ such that $\mathfrak{A}/\mathfrak{N}$ is semi-simple. (See [1], [7].) We say $\mathfrak{A}$ is separable if it is semi-simple and remains so under extensions of the base field (or that the centers of its simple components are separable field extensions of the base field).

Before proceeding to introduce group operators, we recall here the Wedderburn principal theorem, which has been proved for several classes of algebras:

If $\mathfrak{A}$ is an algebra with radical $\mathfrak{N}$ such that $\mathfrak{A}/\mathfrak{N}$ is separable, then $\mathfrak{A}$ contains a subalgebra $\mathfrak{S}$ such that $\mathfrak{A} = \mathfrak{S} + \mathfrak{N}$, $\mathfrak{S} \cap \mathfrak{N} = 0$, $\mathfrak{S} \cong \mathfrak{A}/\mathfrak{N}$.

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