TWISTED RANKS AND EULER CHARACTERISTICS

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In a previous paper [2] the author introduced the notion of the twisted Euler characteristic of a complex on which a group of prime order operates. The twisted Euler characteristic of the complex is equal to that of the fixed-point set; this generalizes part of a classical result of P. A. Smith.

In view of the theorem of Artin and Tate on periodicity in the homology of a finite group [1, XII, 11] the twisted Euler characteristic may be generalized to other groups of operators, and the theorem quoted above remains true if the most generous notion of fixed-point set is adopted. This generalization is made here.

The standpoint is that of the theory of abstract abelian categories (the "exact" categories of Buchsbaum [1, appendix]). Although no application, other than the one just mentioned, is considered here, it is clear that similar constructions may be made, for example, in categories of sheaves.

1. Ranks and Euler characteristics

If $\mathcal{K}$ is an abelian category, a *rank* on $\mathcal{K}$ is a function ρ on the objects of $\mathcal{K}$ with values in an additive group, such that if $0 \to A' \to A \to A'' \to 0$ is exact, then $\rho A = \rho A' + \rho A''$. In particular, then, $\rho 0 = 0$.

For example, on the category of finite dimensional vector spaces over a field, the dimension is a rank. On the category of finite abelian groups $\mathfrak{o}G$, the logarithm of the order of G is a rank.

LEMMA 1. *If ρ is a rank on $\mathcal{K}$ and the diagram*

$$\to A_{2n-1} \to B_{2n-1} \to C_{2n-1} \to \cdots \to B_0 \to C_0 \to \quad (\text{with } C_0 \to A_{2n-1})$$

is exact in $\mathcal{K}$, then

$$\sum_{i=0}^{2n-1} (-1)^i \rho B_i = \sum_{i=0}^{2n-1} (-1)^i \rho A_i + \sum_{i=0}^{2n-1} (-1)^i \rho C_i .$$

For if $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ are the kernels in A_i, B_i, C_i, then, writing indices modulo $2n$:

$$\rho A_j = \rho \bar{A}_j + \rho \bar{B}_j$$

$$\rho B_j = \rho \bar{B}_j + \rho \bar{C}_j$$

$$\rho C_j = \rho \bar{C}_j + \rho \bar{A}_{j-1}$$

from which the lemma follows immediately.

The notation $\mathcal{K}'$ will be used for the category of finitely graded objects

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