ON THE CHARACTERISTIC LINEAR SYSTEMS OF ALGEBRAIC FAMILIES

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The completeness of the characteristic linear systems of algebraic families is one of the important and interesting problems in algebraic geometry. Recently it became clear that this is not true in general in the case of prime characteristic [1], but it is still an interesting question to ask what conditions are necessary or sufficient for the completeness of the characteristic systems. O. Zariski proposed to me the problem of extending to the abstract case the result that the vanishing of the geometric genus is sufficient for this purpose. This paper contains the affirmative answer to his conjecture. We shall prove in this paper the following: On the nonsingular surface we have an inequality $p_g \geqq h^{0,1} - q \geqq 0$ where p_g, $h^{0,1}$, and q denote respectively the geometric genus, maximal deficiency, and the irregularity (the dimension of the Picard variety) of the original surface. In §1 we shall introduce the notion of the characteristic set of an algebraic family and prove the linearity of that set. Then we shall show the inequality $q \leqq h^{0,1}$ for the nonsingular variety of any dimension (§2). In §3 we shall prove an important property of an ample linear system on the nonsingular variety, and then the final results can be deduced from this property in the case of the nonsingular surfaces. The author was inspired very much by the works of F. Severi[1] during these researches.

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I. The characteristic linear systems

Let V be a projective variety of dimension $\geqq 2$ which is irreducible and free from singular subvarieties of codimension 1.[2] Let Σ be a maximal algebraic family of positive divisors on V, and let k be a common field of definition for V and Σ. We shall assume in the following that the generic member C of Σ over k is an irreducible variety and that any subvariety of C of codimension 1 is simple, not only on C, but also on V. Let C_0 be a fixed generic member of Σ over k, and C a generic member of Σ over $k(C_0) = k'$.[3] Since $C \succ\!\!\!\!/\, C_0$, we

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[1] Cf. Zariski [14], pp. 82–84.

[2] The term "variety" is used only for an irreducible variety, and the terms "irreducible", "simple", and "normal" in this paper are used always in the absolute sense.

[3] We shall mean by $k(C)$ the smallest field containing k over which C is rational. This is equivalent to the field generated over k by the Chow point of C, since C is a V-divisor and V is defined over k.