MAPPINGS ON S^1 INTO ONE-DIMENSIONAL SPACES

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Throughout this paper, P is the cartesian plane, S^1 is the unit circle in P, D is the closed disk in P whose boundary is S^1, and Y is a metric space. Two theorems concerning homotopy properties of mappings on S^1 into Y are proved in this paper.

LEMMA 1. *If $f: S^1 \rightarrow Y$ is inessential, then there exists a continuous extension $F: D \rightarrow Y$ of f such that none of the components of the inverse sets $F^{-1}(y)$, $y \in Y$, separates the plane P.*

Proof. Let $f: S^1 \rightarrow Y$ be inessential. Then, there exists a continuous extension $g: D \rightarrow Y$ of f. We define A to be the set of all components of sets $g^{-1}(y)$, $y \in Y$. It is well known that A is an upper semicontinuous decomposition of D. If a and b are members of A, we define $a < b$ to mean that a is contained in a bounded component of $P - b$.

Let a, b, and c be members of A, and suppose $a < b$ and $b < c$. Then, $a \subset \beta$ and $b \subset \gamma$, where β and γ are bounded components of $P - b$ and $P - c$ respectively. Since c is connected, γ must be simply connected. It follows that β, being a bounded component of $P - b$, is contained in γ. Thus $a \subset \gamma$ and $a < c$. This proves that $<$ partially orders A.

Next, suppose that a, b_1, and b_2 are members of A, $a < b_1$, and $a < b_2$. We will prove that either $b_1 \leqq b_2$ or $b_2 \leqq b_1$. There exist bounded components β_1 and β_2 of $P - b_1$ and $P - b_2$ respectively such that $a \subset \beta_1$ and $a \subset \beta_2$. If β_1 and β_2 have a common boundary point p, then $p \in b_1 \cap b_2$ and $b_1 = b_2$. Otherwise, either $\bar{\beta}_1 \subset \beta_2$ or $\bar{\beta}_2 \subset \beta_1$. In the first case, $b_1 \subset \beta_2$ and $b_1 < b_2$, and in the second case, $b_2 \subset \beta_1$ and $b_2 < b_1$.

Let $a \in A$. By Zorn's lemma, there exists $b \in A$ such that $a \leqq b$ and b is maximal with respect to the relation $<$. It follows from the result proved in the preceding paragraph that such a maximal b must be unique.

Now let $x \in D$. There is an element $a \in A$ such that $x \in a$, and there is a unique maximal $b \in A$ such that $a \leqq b$. We define $F(x) = g[b]$. We must show that F has the desired properties.

For $a \in A$, we define a^* to be the union of a and the bounded components of $P - a$. Each such a^* is a continuum which does not separate the plane. The set $\{F^{-1}(y) \mid y \in Y\}$ is seen to be the same as $\{a^* \mid a \text{ maximal in } A\}$. This latter set is easily proved to be upper semicontinuous, since A is, and hence F is continuous. If $x \in S^1$ and $x \in a \in A$, then a is maximal in A, and $F(x) = g[a] = g(x) = f(x)$. This proves that F is an extension of f.

A *dendrite* is a locally connected continuum which does not contain a simple closed curve. A space is *contractible* if and only if the identity mapping of

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