SOME EXTREME VALUE RESULTS FOR INDEFINITE HERMITIAN MATRICES[1]

BY M. MARCUS, B. N. MOYLS, AND R. WESTWICK

1. Introduction

Let A be an n-square complex Hermitian matrix, and let $x_1, \cdots, x_k$ be an orthonormal (o.n.) set of vectors in the unitary n-space V_n. In this paper we consider the following two functions:

$$\varphi(x_1, \cdots, x_k) = \prod_{j=1}^{k} (Ax_j, x_j), \tag{1.1}$$

$$\psi(x_1, \cdots, x_k) = E_2((Ax_1, x_1), \cdots, (Ax_k, x_k)), \qquad k \leqq n. \tag{1.2}$$

$E_2(y_1, \cdots, y_k)$ is the second elementary symmetric function of the indicated variables. The problem is to determine the extreme values of the functions φ and ψ as the vectors $x_1, \cdots, x_k$ vary in V_n subject to the restriction

$$(x_i, x_j) = \delta_{ij}.$$

To do this, we examine the structure of extremal sets $x_1, \cdots, x_k$ in terms of invariance under A. We shall consistently use the term "extremal set" to denote a set of extremal vectors, i.e., vectors for which the extreme values of φ and ψ occur. The problem of the minimum for φ when A is nonnegative Hermitian has been solved by K. Fan [4] and later generalized by Amir-Moéz [1]. The maxima for both φ and ψ are contained in [7]. The minimum for ψ, again with A nonnegative Hermitian, has been solved by A. Ostrowski by means of Schur-convex functions [8]. In this paper we will assume that A *has both positive and negative eigenvalues.* The usual techniques do not seem to generalize readily from the case of positive matrices.

2. Invariance results

LEMMA 1. *If A is nonsingular, then an extremal set for φ spans a k-dimensional invariant subspace of A.*

Proof.[2] By the continuity of the inner product it is clear that we may select $y_1, \cdots, y_k$ satisfying $(y_i, y_j) = \delta_{ij}$ such that

$$\min \varphi = \varphi(y_1, \cdots, y_k). \tag{2.1}$$

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[2] This proof is similar to that of Lemma 1 of [7] in which A is assumed positive definite Hermitian.