PAIRS OF MATRICES OF ORDER TWO WHICH GENERATE FREE GROUPS[1]

BY K. GOLDBERG AND M. NEWMAN

Throughout this paper $A = (a_{ij})$ and $B = (b_{ij})$ will denote rational integral unimodular matrices of order two which are not of finite period.

Let us say that an element of a matrix is *dominant* if it is larger in absolute value than any other element of the matrix.

Our object is to prove the following theorem:

THEOREM. *If a_{12} is dominant in A and b_{21} is dominant in B, then A and B generate a free group.*

The first result in this direction was due to I. N. Sanov [1] who proved that $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and A^T generate a free group. The methods used in this paper are derived from Sanov's proof of his result.

More recently J. L. Brenner [2] has shown that $A = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and A^T generate a free group for all real $m \geqq 2$.

These results were brought to our attention by Professor Brenner and a generalization was suggested by O. Taussky-Todd.

1. Two lemmas

We find it convenient to separate the proof of the theorem into two parts which are described by the lemmas below.

We define $A^n = (a_{ij}^{(n)})$ and $B^n = (b_{ij}^{(n)})$ where n is an integer.

LEMMA 1. *If $a_{12}^{(n)}$ is dominant in A^n and $b_{21}^{(n)}$ is dominant in B^n for all $n \neq 0$, then A and B generate a free group.*

LEMMA 2. *If a_{12} is dominant in A, then $a_{12}^{(n)}$ is dominant in A^n for all $n \neq 0$.*

If A has trace t and determinant d, then the fact that A is not of finite period is used only to imply that $t \neq 0$ for $d = -1$ and $|t| \geqq 2$ for $d = 1$.

The fact that a_{12} is dominant in A implies $|a_{12}| -2$, $|a_{11}a_{22}| -1$, $|a_{11}| - |a_{21}|$ and $|a_{12} - a_{11}| - |a_{22} - a_{21}|$ are all nonnegative: $|a_{12}|$ is at least 2 because at least one other element is not 0, neither diagonal element vanishes because then $|a_{12}| > 1$ would divide the determinant $d = \pm 1$, a_{21} is the least element because $|a_{11}a_{22} - a_{12}a_{21}| = 1$ and $|a_{12}| - |a_{ii}| \geqq 1$, and

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