A JACOBIAN CRITERION OF SIMPLE POINTS

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Previously Zariski [5] gave a Jacobian criterion of simplicity of points of an algebraic variety, and its generalization to the algebroid case[1] was treated by Samuel [4].[2] In the present paper, we shall give a proof of the criterion of simplicity. Although we shall treat the algebroid case, our proof is also valid for the algebraic case if formal power series rings are replaced by polynomial rings.

1. Derivations of a ring (cf. [3])

Let $\mathfrak{o}$ be a ring.[3] A *derivation* D of $\mathfrak{o}$ is an additive endomorphism of the total quotient ring L of $\mathfrak{o}$ which satisfies the following conditions: (1) $D(xy) = xDy + yDx$ for $x, y \in L$, (2) there exists an element d of $\mathfrak{o}$ which is not a zero-divisor such that $dDx \in \mathfrak{o}$ for $x \in \mathfrak{o}$. Here, if d can be chosen to be 1, we call D an *integral derivation* of $\mathfrak{o}$.

A derivation D of $\mathfrak{o}$ such that $D\mathfrak{o}' = 0$, $\mathfrak{o}'$ being a subring of $\mathfrak{o}$, is called a derivation over $\mathfrak{o}'$; if $D\mathfrak{o} = 0$, then we say that D is the zero derivation or the trivial derivation of $\mathfrak{o}$, and we denote it by 0.

The set of derivations of $\mathfrak{o}$ over a subring $\mathfrak{o}'$ is an L-module, which will be denoted by $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$. Obviously $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ is generated by integral derivations. Linear dependence of derivations will always mean dependence in this module, hence over L, equivalently over $\mathfrak{o}$. The length of the module $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ (as an L-module) is called the *dimension* of $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ and is denoted by $\dim \mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ (the dimension of $\mathfrak{D}_{\mathfrak{o}/\mathfrak{o}'}$ may be infinite).

Let $\mathfrak{a}$ be an ideal of $\mathfrak{o}$ and let ϕ be the natural homomorphism from $\mathfrak{o}$ onto $\mathfrak{o}/\mathfrak{a}$. Let D be a derivation of $\mathfrak{o}$. Assume that there exists an element $d \in \mathfrak{o}$ which is not a zero-divisor modulo $\mathfrak{a}$ and is such that (i) dD is an integral derivation of $\mathfrak{o}$ and (ii) $dD\mathfrak{a} \subseteqq \mathfrak{a}$. Then we can define an operator D' in $\mathfrak{o}/\mathfrak{a}$ to be $D'(\phi(x)) = \phi(dDx)/\phi(d)$ $(x \in \mathfrak{o})$. D' can be uniquely extended to a derivation of $\mathfrak{o}/\mathfrak{a}$ (independently of the choice of d). The derivation obtained in this manner is called the derivation *induced* in $\mathfrak{o}/\mathfrak{a}$ by D.

2. Derivations of a local ring

LEMMA 1. *Let $\mathfrak{o}$ be a local ring with maximal ideal $\mathfrak{m}$, $\mathfrak{o}'$ a subring of $\mathfrak{o}$, and let $f_1, \cdots, f_r$ be a set of generators of $\mathfrak{m}$. Assume that a subset M of $\mathfrak{o}$ generates*

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[1] The notion of algebroid varieties over an algebraically closed field was introduced by Chevalley [1]. An algebroid variety can be defined similarly over an arbitrary field.

[2] Samuel's argument seems to be valid only if $[k:k^p]$ is finite. His treatment of the general case being too sketchy, we prefer to give a proof based upon other methods.

[3] A ring means a commutative ring with identity.