UNDERPOLYNOMIALS AND INFRAPOLYNOMIALS[1,2]

BY T. S. MOTZKIN AND J. L. WALSH

Introduction

If E is a point set of the z-plane containing at least n points, a polynomial $g(z) \equiv z^n + g_1 z^{n-1} + \cdots + g_n$ is called an *underpolynomial* of $f(z) \equiv z^n + f_1 z^{n-1} + \cdots + f_n$ on E provided we have $g(z) \not\equiv f(z)$ and

$$(1) \qquad | g(z)| < | f(z)| \qquad \text{on } E \text{ where } f(z) \neq 0,$$

$$(2) \qquad g(z) = f(z) \qquad \text{on } E \text{ where } f(z) = 0.$$

The polynomial $f(z)$ is called an *infrapolynomial* on E if it has no underpolynomials on E, a concept due to Fekete and von Neumann [1]. Infrapolynomials as such have been studied also by Fekete [1] and the present writers [3].

The importance of infrapolynomials lies primarily in the fact that a polynomial of the form $f(z) \equiv z^n + \cdots$ which minimizes (among all polynomials of that form) one of the classical norms ($p > 0$)

$$(3) \qquad \sup [\, | f(z) |, z \text{ on } E],$$

$$(4) \qquad \int_E | f(z) |^p \, | dz |,$$

$$(5) \qquad \iint_E | f(z) |^p \, dS,$$

must clearly be an infrapolynomial on E; of course for (4) or (5) to have a meaning, E must be rectifiable or have positive area. The extremal polynomials with norms (4) and (5), $p = 2$, are orthogonal on E, hence particularly important; they include the widely studied Legendre, Tchebycheff, and Jacobi polynomials if suitable weight functions are introduced.

If a set E consists of $n + 2$ points, an arbitrary function $F(z)$ to be approximated on E by a polynomial of degree n can be replaced on E by an equal polynomial $P(z)$ of degree $n + 1$, so the problem of best approximation to $F(z)$ on E is essentially the problem of studying the polynomial $z^{n+1} + \cdots$ of least norm on E [compare Motzkin and Walsh, 1, §8].

The object of the present paper is to investigate systematically the properties of the class of infrapolynomials of given degree on a bounded set. The strong inequality is important in (1) in its effect on the norm (3) but not

Received August 20, 1956.

[1] This paper was presented to the American Mathematical Society, August 24, 1956.

[2] This research was sponsored in part by the Office of Naval Research, U. S. Navy, the Office of Ordnance Research, U. S. Army, and by the Office of Scientific Research, Air Research and Development Command.