SMOOTHING METHODS FOR CONTOURS

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1. Introduction

The method of using contours in the investigations of problems involving surface area and the calculus of variations was introduced by L. Cesari [3] and has been used by both of the present authors in developing the theory of surfaces from a point of view which differs in many respects from the classical point of view in that it depends less on analytical techniques. The method of contours uses chiefly topological ideas and hence would appear to be more closely related to the true nature of a surface which is essentially topological. We shall describe here how contours are defined and give several methods of smoothing contours so that certain types of problems in surface theory can be more conveniently treated.

Let J be a simply connected Jordan region in the plane, and let $T: J \to E_3$ be a continuous map. T defines a surface S under the standard Fréchet definition. Although a theory of contours can be developed for multiply connected regions or even 2-manifolds, we shall be concerned in this paper only with simply connected Jordan regions. If $[S]$ is the set of points in E_3 occupied by the surface, let f be any real valued continuous function defined in E_3, and if $p = T(w)$, $w \in J$, $p \in E_3$, we define $F: J \to$ reals by $F(w) = f(T(w))$. For any real value of t, $-\infty < t < \infty$, define $C(t)$ to be the subset of J for which $F(w) = t$, $D^-(t) = \{w \in J \mid F(w) < t\}$, $D^+(t) = \{w \in J \mid F(w) > t\}$. (Some of these sets may be empty.) Evidently $C(t)$ is compact, and $D^-(t)$, $D^+(t)$ are open in J for all t and have their boundaries contained in $C(t)$. For a fixed value of t let $\{\alpha\}$ be the collection of all components of $D^-(t)$ and for each α let $\{\gamma\}_\alpha$ be the family of all components of the set $\bar{\alpha} - \alpha \subset C$. If we denote the union of all the γ for all $\alpha \in \{\alpha\}$ by $\kappa(t)$, we say that $\kappa(t)$ is the *contour* associated with the value t for the mapping T and the function f. Since for the same surface S, different representations T and different functions f may be considered, it can be seen that a large variety of types of contours can exist. R. E. Fullerton [6] has shown that if S is nondegenerate, a representation T of S can be chosen in such a way that a countable dense set of the contours are sums of arcs and simple closed curves, provided that S has finite area. Other authors [1, 2, 3, 5, 8, 9] have consistently used contours in questions arising from the calculus of variations for surfaces. Cesari [3] has defined a generalized length for the image $T(\kappa)$ in the following manner. Let $\gamma \in \{\gamma\}_\alpha$ for some $\alpha \in \{\alpha\}$, and let $A(\gamma, \alpha)$ be the set of all points $w \in J$ for which either $w \in \alpha$ or w is separated from γ by other components $\gamma' \in \{\gamma\}_\alpha$. Let $\{\eta\}_{A,\gamma}$ be the set of all ends of $A(\gamma, \alpha)$ ending on γ, and let $\{\eta_i\}$, $i = 1, 2, \cdots, n$, be any finite set for which $\eta_i < \eta_j$

Received August 7, 1956.