SOLVABLE FACTORIZABLE GROUPS

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Let H and K be subgroups of a finite group G, and let $G = HK$. During recent years, a number of theorems of the following type have been proved (see [2], [3], [4], [5]): if H and K satisfy certain conditions, then G is solvable. In this paper, four additional theorems of this kind will be given. It will be shown that G is solvable under any of the following conditions: (i) H nilpotent, K Abelian or Hamiltonian; (ii) H nilpotent of odd order, K contains a subgroup L of index 2 such that all subgroups of L are normal in K; (iii) H cyclic, K contains a subgroup L of index 2 or 3 such that all subgroups of L are normal in K; (iv) H dihedral or dicyclic, K dihedral or dicyclic.

The proofs of these theorems follow the pattern of earlier proofs by Huppert and Itô [2], [3], [4], [5]. Part (i) generalizes a theorem of Itô [5] for the case where K is Abelian. Part (ii) should be compared with the theorem of Huppert and Itô [4, Satz 4] where H is only nilpotent, but L is cyclic (all subgroups of L being automatically normal in K in this case). Part (iv) is a generalization of a theorem of Huppert [2, Satz 3] that G is solvable if H and K are both dihedral. Its proof requires a theorem (Theorem 4) concerning the primitive S-rings of Schur and Wielandt (see [6], [8], [9]), to the effect that there are no primitive S-rings over a generalized dicyclic group. Finally, it should be mentioned that a generalization (Theorem 5) of (iv) is proved. The extent of this generalization depends on knowledge of permutation groups not yet available.

All groups are to be finite. The following notation will be used: $H \subset G$, $H < G$, $H \lhd G$, to denote that H is a subgroup, proper subgroup, or normal subgroup of G respectively, G_p for a Sylow p-subgroup of G, ${}_pG$ for a Sylow p-complement of a nilpotent group G, $G'(p)$ for the p-commutator subgroup, $Z(G)$ for the center of G, $N(H)$ for the normalizer of H in G, $o(G)$ for the order of G, $[G:H]$ for the index of H in G, H^g for $g^{-1}Hg$, and $\{A, \cdots\}$ for the subgroup generated by $A, \cdots$.

THEOREM 1. *If $G = HK$, where H is nilpotent and K is Abelian or Hamiltonian, then G is solvable.*

Proof. Any subgroup or factor group of an Abelian or Hamiltonian group is again Abelian or Hamiltonian. The proof in [5] now applies without change.

THEOREM 2. *If $G = HK$, where H is nilpotent of odd order, and K has a subgroup L of index 2 such that every subgroup of L is normal in K, then G is solvable.*

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