A PROBABILISTIC APPROACH TO PROBLEMS OF DIOPHANTINE APPROXIMATION

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Introduction

Let $z_1, z_2, \cdots, z_n$ denote unimodular complex numbers

$$|z_j| = 1, \qquad j = 1, 2, \cdots, n.$$

We put $z_j = e^{i\varphi_j}(0 \leqq \varphi_j < 2\pi)$ and

$$S_k = \sum_{j=1}^{n} z_j^k \qquad (k = 1, 2, \cdots). \tag{1}$$

By a well known theorem of *Dirichlet*, for any integer $\omega \geqq 2$ we can find a positive integer k with $1 \leqq k \leqq \omega^n$ and integers $b_1, b_2, \cdots, b_n$ such that

$$\left| \frac{k\varphi_j}{2\pi} - b_j \right| \leqq \frac{1}{\omega} \qquad (j = 1, 2, \cdots, n). \tag{2}$$

It follows for $\omega \geqq 5$ that among the power sums S_k $(1 \leqq k \leqq \omega^n)$, there is at least one for which

$$|S_k| \geqq n \cos \frac{2\pi}{\omega}.$$

This can be stated also as follows: *For any choice of the unimodular numbers* z_j $(j = 1, 2, \cdots, n)$, *we have*

$$\operatorname*{Max}_{1 \leqq k \leqq [A(c)]^n} |S_k| \geqq cn \tag{3}$$

for any c such that $0 < c < 1$, *where* $A(c) = [2\pi/\arccos c] + 1$. (Here and in what follows $[x]$ denotes the integral part of x.)

It is well known that Dirichlet's theorem can not be improved. For instance, if $\varphi_j = 2\pi/\omega^j$ $(j = 1, 2, \cdots, n)$, where $\omega \geqq 2$ is an integer, then among the integers $1 \leqq k \leqq \omega^n - 1$ there is none for which all the inequalities

$$\left| \frac{k\varphi_j}{2\pi} - b_j \right| < \frac{1}{\omega} \qquad (j = 1, 2, \cdots, n),$$

where $b_1, b_2, \cdots, b_n$ are integers, would be satisfied.

A simple example of *G. Hajós* (see [1], p. 16) shows that Dirichlet's theorem can not be much improved, even when we admit nonintegral values for k. The example of *Hajós* is as follows: if we choose

$$\varphi_j = \frac{2\pi}{6 \cdot 5^{j-1}} \qquad (j = 1, 2, \cdots, n),$$

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