ON SOME APPLICATIONS OF DYNAMIC PROGRAMMING TO MATRIX THEORY

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1. Introduction

The purpose of this paper is to discuss some applications of the functional equation technique of dynamic programming to some questions of matrix theory.

We shall first consider the solution of a system of linear equations,

$$Ax = b, \tag{1.1}$$

where A is a Jacobi matrix. Then we shall discuss the same problem for the case where A is "almost" a block-diagonal matrix. Matrices of this type arise in the study of weakly coupled mechanical or electrical systems. Finally, we shall discuss the calculation of the largest or smallest characteristic values of matrices of this type.

2. Jacobi matrices

There is a large body of literature connected with systems of linear equations of the form

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 &= b_1\,, \\ a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 &= b_2\,, \\ &\vdots \\ a_{N,N-1}\,x_{N-1} + a_{N,N}\,x_N &= b_N\,. \end{aligned} \tag{2.1}$$

If $a_{ij} = a_{ji}$, the associated matrix A is called a Jacobi matrix. Assuming that A is positive definite, we wish to obtain the solution of this system in a form quite different from any of the solutions furnished by classical methods.

3. Functional equations

If A is positive definite, the solution of the system (2.1) is equivalent to determining the minimum of the inhomogeneous form

$$Q(x) = \sum_{i,j=1}^{N} a_{ij}\,x_i\,x_j - 2\sum_{i=1}^{N} b_i\,x_i\,. \tag{3.1}$$

Let us define the auxiliary sequence of functions

$$f_k(z) = \operatorname{Min}_{\{x\}} \left[\sum_{i,j=1}^{k} a_{ij}\,x_i\,x_j - 2\sum_{i=1}^{k-1} b_i\,x_i - 2zx_k\right], \tag{3.2}$$

$k = 1, 2, \cdots, N$, $-\infty < z < \infty$. We wish to determine $f_N(b_N)$ and the point $[x_1, x_2, \cdots, x_N]$ at which the minimum is attained. It is easy to see

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