ON INGHAM'S TRIGONOMETRIC INEQUALITY

BY L. J. MORDELL

Ingham has recently[1] proved the following

THEOREM. *Let*

$$f(t) = \sum_{n=N}^{N'} a_n e^{-\lambda_n t i},$$

where the λ*'s are real and* $\lambda_n - \lambda_{n-1} \geqq \gamma > 0$ $(N < n \leqq N')$, *and let* $\gamma T = \pi$. *Then*

$$(1) \qquad | a_n | \leqq \frac{1}{T} \int_{-T}^{T} | f(t) | \, dt \qquad (N \leqq n \leqq N').$$

He notes that we may take $\gamma = 1$, $T = \pi$ by the substitution $\gamma t = t'$. We may then rewrite the result as the

THEOREM. *Let*

$$(2) \qquad f(t) = \sum_{r=0}^{n} a_r e^{-\lambda_r t i},$$

where the λ*'s are real and* $\lambda_r - \lambda_{r-1} \geqq 1$ $(1 \leqq r \leqq n)$. Then

$$(3) \qquad | a_r | \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} | f(t) | \, dt \qquad (0 \leqq r \leqq n).$$

His proof, to which he was led by considerations of Fourier transforms, is quite short. Its essential idea, however, as I see it, can be presented in a rather simpler way, which also leads to a more precise result. He has shown that the factor $1/T$ in (1) cannot be replaced by a factor c/T where c is an absolute constant <1, but my proof shows that the factor $1/\pi$ in (3) can be replaced by a factor $K_r < 1/\pi$ depending upon the λ's.

On multiplying (2) throughout by $e^{\lambda_r t i}$, it suffices to take $f(t)$ in the form

$$(4) \quad f(t) = \sum_{r=-m}^{n} a_r e^{-\lambda_r i t}, \quad \lambda_0 = 0, \qquad \lambda_r - \lambda_{r-1} \geqq 1 \quad (-(m-1) \leqq r \leqq n),$$

and to estimate $| a_0 |$. I prove that

$$(5) \qquad | a_0 | \leqq \frac{K_0}{\pi} \int_{-\pi}^{\pi} | f(t) | \, dt,$$

with

$$(6) \qquad K_0 = 1 - \frac{1}{2} \prod_{r=-m}^{r=n}{}' (\mu_r/\lambda_r),$$

Received November 6, 1956.

[1] *A further note on trigonometric inequalities*, Proceedings of the Cambridge Philosophical Society, vol. 46 (1950), pp. 535–537.