THE IRREDUCIBLE REPRESENTATIONS OF A SEMIGROUP RELATED TO THE SYMMETRIC GROUP

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1. Introduction

A. H. Clifford [2] has studied the representations of a class of semigroups. His results lead to a complete classification of the representations of a particular class of semigroups having considerable independent interest. These semigroups are the semigroups $\mathfrak{T}_n$ defined as follows.

Consider a finite set consisting of say n elements; for the sake of definiteness we may consider the set $\{1, 2, \cdots, n\}$. Let $\mathfrak{T}_n$ be the set of all single-valued mappings of this set onto or into itself. For $f, g \in \mathfrak{T}_n$ let fg be the element of $\mathfrak{T}_n$ such that $fg(i) = f(g(i))$ $(i = 1, \cdots, n)$. With this definition of multiplication, $\mathfrak{T}_n$ is obviously an associative system, *i.e.*, a semigroup. The order of $\mathfrak{T}_n$ is n^n; $\mathfrak{T}_n$ contains the symmetric group $\mathfrak{S}_n$, properly if $n > 1$; $\mathfrak{T}_n$ is noncommutative if $n > 1$.

By the term (α, β) matrix, we shall mean a matrix with α rows and β columns and complex entries. A representation of a semigroup G is a homomorphism M of G into the multiplicative semigroup of all (α, α) matrices (α an arbitrary positive integer) such that $M(x) \neq 0$ for some $x \in G$. If the set $\{M(x)\}_{x \in G}$ is an irreducible set of matrices (*i.e.*, if every (α, α) matrix is a linear combination of matrices $M(x)$), then M is said to be an irreducible representation of G. The identity representation is the mapping that carries every $x \in G$ into the identity matrix.

In the present paper we give an explicit determination of all irreducible representations of $\mathfrak{T}_n$. The idea of studying $\mathfrak{T}_n$ was suggested to us by D. D. Miller (oral communication). The problem of obtaining representations of semigroups as distinct from groups seems to have been first studied by Suškevič [6]. A. H. Clifford [2] has, as noted above, given a construction of all representations of a class of semigroups closely connected with $\mathfrak{T}_n$. Ponizovskiĭ: [5] has pointed out some simple properties of $\mathfrak{T}_n$. In the present paper we also relate the irreducible representations of $\mathfrak{T}_n$ to the semigroup algebra $\mathfrak{L}_1(\mathfrak{T}_n)$ (notation as in [3]).

2. Definitions

Let f be an element of $\mathfrak{T}_n$. Then f splits the set $\{1, 2, \cdots, n\}$ into a number, p, of nonvoid, disjoint subsets, each of the form $\{x : f(x) = a\}$ for some a in the range of f. Obviously f is determined by these sets and the corresponding a's. We will set down a unique notation for the elements

Received October 18, 1956.

[1] Research supported in part by the National Science Foundation. The first named author is a fellow of the John Simon Guggenheim Memorial Foundation.