ON MATRIX CLASSES CORRESPONDING TO AN IDEAL AND ITS INVERSE

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1. It is known (Latimer and MacDuffee [1], Taussky [2], Zassenhaus [3], Reiner [4]), that there is a 1–1 correspondence between classes of $n \times n$ matrices A of rational integers and ideal classes. The matrix A is assumed to be a zero of an irreducible polynomial $f(x)$ of degree n with rational integral coefficients and first coefficient 1. The class associated with A consists of all matrices $S^{-1}AS$ where S runs through all unimodular matrices with rational integral coefficients. Let α be an algebraic number root of $f(x) = 0$. Then the 1–1 correspondence between the matrix classes and the ideal classes may be described as follows: If $(\alpha_1, \cdots, \alpha_n)$ is a modular basis for an ideal $\mathfrak{a}$ in the ring generated by α and $\alpha(\alpha_1, \cdots, \alpha_n)' = A(\alpha_1, \cdots, \alpha_n)'$, then the ideal class determined by $\mathfrak{a}$ corresponds to the matrix class determined by A. In what follows we assume that the numbers $1, \alpha, \alpha^2, \cdots$ form an integral basis in the field $R(\alpha)$.

It was further shown (Taussky [5], [6]) that for quadratic fields the matrix class generated by the transpose of A corresponds to the inverse class. It is now shown that this is always true. This fact is established in two different ways, once directly, secondly by using a known lemma (Hasse [7], pp. 327–328). Both proofs make use of the so-called complementary ideal (see Dedekind [8], pp. 374–376; see also Hecke [9], pp. 131–133).

It is easily seen directly that both the companion matrix C of $f(x)$ and its transpose correspond to the principal class in $R(\alpha)$. Hence

$$C' = S^{-1}CS$$

where S is unimodular. The matrix S can be constructed explicitly.

It is further shown that the matrix classes defined by unimodular matrices S with $|S| = 1$ coincide with the classes defined by $|S| = \pm 1$ if and only if the field has a unit ε with norm $\varepsilon = -1$.

In [5], [6] the matrix classes which correspond to ideal classes of order 2 in a quadratic field were studied. The transpose of a matrix in such a class belongs to the same class. It is now shown that such a class contains a symmetric matrix if the fundamental unit ε has norm $\varepsilon = -1$. This can also be regarded as a special case of a theorem proved by Faddeev [10] from a different point of view.

2. THEOREM 1.[2] *Let the matrix A correspond to the ideal class determined*

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