ON A PROBLEM OF PICARD CONCERNING SYMMETRIC COMPOSITUMS OF FUNCTION-FIELDS[1]

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1. Introduction

Let K be a fixed algebraically closed ground-field of arbitrary characteristic. Function-fields and varieties will be considered over K. Function-fields of Abelian varieties will be called Abelian function-fields. Let Σ be a function-field of dimension r, and let $\Sigma(m)$ be its m-fold symmetric compositum, i.e., the invariant subfield of the m-fold direct compositum of Σ under the symmetric group of permutations of factors. Obviously, $\Sigma(1)$ is an Abelian function-field if and only if Σ is so. Moreover, $\Sigma(m)$ is an Abelian function-field if and only if m is the genus of Σ in case $r = 1$. In this paper, *we shall show that $\Sigma(m)$ can never become an Abelian function-field for r, $m > 1$.* This fact was already remarked by Picard in the case $r = 2$.[2] His reasoning applies to the case of even r, but, as he himself observed, not directly to the case of odd r. Thus, our result includes the case which Picard failed to discuss.

2. Reduction of the problem

Let V be a projective model of Σ, and let U and $V(m)$ be the m-fold direct and symmetric products of V. Then, there is a canonical rational map from U to $V(m)$, and $\Sigma(m)$ is the function-field of $V(m)$. Moreover, V and $V(m)$ have the same Albanese variety, say A. In fact, let p_i be the projection of U to its i^{th} factor for $i = 1, \cdots, m$; let f be a canonical map of V to its Albanese variety A. Then, $F = \sum_{i=1}^{m} f \circ p_i$ is the product of the canonical rational map from U to $V(m)$ and a canonical map of $V(m)$ to A. The converse is also true.[3] On the other hand, if we replace V by the graph of f, we can assume that f is regular on V. Furthermore, if we replace V by its derived normal model, we can assume, in addition, that V has negligible singularities, i.e., that the singular locus of V is of co-dimension at least equal to 2.[4] In this case, U has also negligible singularities.

Now, assume that $\Sigma(m)$ is an Abelian function-field for some r, $m > 1$. Let A be an Abelian variety such that $\Sigma(m)$ is the corresponding function-field. Then, A is the Albanese variety of $V(m)$, and a canonical map of $V(m)$

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[2] Cf. E. PICARD AND G. SIMART, *Théorie des fonctions algébriques de deux variables indépendentes*, 2, Paris, 1906, pp. 469–474. The problem is raised on p. 474.

[3] These are immediate consequences of the definition of Albanese varieties and of the Corollary on p. 32 of A. WEIL, *Variétés abéliennes et courbes algébriques*, Paris, 1948.

[4] The passage from V to the derived normal model of the graph of f is a standard process introduced by Zariski. Cf., *Foundations of a general theory of birational correspondences*, Trans. Amer. Math. Soc., vol. 53 (1943), pp. 490–542.