INEQUALITIES FOR ASYMMETRIC ENTIRE FUNCTIONS[1]

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Let $p_n(z)$ be a polynomial of degree n such that $|p_n(z)| \leqq 1$ in the unit disk $|z| \leqq 1$. The following results are well known.

THEOREM A. *For* $|z| = R > 1$, $|p_n(z)| \leqq R^n$.

THEOREM B. *For* $|z| = 1$, $|p_n'(z)| \leqq n$.

Theorem A is a simple deduction from the maximum principle (see [11], p. 346, or [10], vol. 1, p. 137, problem III 269). Theorem B is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [12], or [2], pp. 206, 231).

When $p_n(z)$ has no zeros in $|z| < 1$, more precise statements can be made:

THEOREM C. *For* $|z| = R > 1$, $|p_n(z)| \leqq \frac{1}{2}(1 + R^n)$.

THEOREM D. *For* $|z| = 1$, $|p_n'(z)| \leqq \frac{1}{2}n$.

Theorem D was conjectured by Erdös and proved by Lax [8]; for another proof see [4]. Theorem C was deduced from Theorem D by Ankeny and Rivlin [1].

Since $p_n(e^{iz})$ is an entire function of exponential type, these theorems suggest generalizations to such functions. Let $f(z)$ be an entire function of exponential type τ, with $|f(x)| \leqq 1$ for real x.

THEOREM A′. *For all* y, $|f(x + iy)| \leqq e^{\tau|y|}$.

THEOREM B′. *For all real* x, $|f'(x)| \leqq \tau$.

Theorem A′ is a simple consequence of the Phragmén-Lindelöf principle (for references see [2], p. 82; see also [11], pp. 346–347). Theorem B′ is Bernstein's generalization of Theorem B (see references on Theorem B).

In this note I obtain theorems for entire functions which generalize Theorems C and D. To see what to expect, note that $p_n(e^{iz})$ is an entire function $f(z)$ of exponential type of a special kind: if $h(\theta)$ is its indicator, we have $h(-\pi/2) = n$, but $h(\pi/2) > -n$ unless $p_n(z) = cz^n$. If $p_n(z)$ has no zeros in $|z| < 1$, $f(z)$ has no zeros in $y > 0$, and moreover (since $p_n(0) \neq 0$) $h(\pi/2) = 0$.

Let us consider, then, entire functions $f(z)$ of exponential type τ with $|f(x)| \leqq 1$ for real x, $h(\pi/2) = 0$ (hence necessarily $h(-\pi/2) = \tau$), and $f(z) \neq 0$ for $y > 0$.

THEOREM 1. *For* $y < 0$, $|f(x + iy)| \leqq \frac{1}{2}(e^{\tau|y|} + 1)$.

THEOREM 2. *For all real* x, $|f'(x)| \leqq \frac{1}{2}\tau$.

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