HOMOLOGY OF NOETHERIAN RINGS AND LOCAL RINGS

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Introduction

This paper contains a collection of results on the homology of a residue class ring R/M of a commutative Noetherian ring R, as R-module. More important than the individual results is the general method by which they are obtained, namely, the systematic use of skew-commutative graded differential algebras (called R-algebras in this paper, cf. §1). The functor

$$\mathrm{Tor}^R(R/M,\ R/N)$$

has naturally the structure of an R-algebra (cf. §5), so why not exploit this fact? We show in §2 that it is always possible to construct a free resolution of R/M which is an R-algebra, and in §3 and §4, we show that in some important cases our abstract method of construction yields a concrete efficient resolution (Theorem 4). Our "adjunction of variables" is a naive approach to the exterior algebras and twisted polynomial rings familiar to topologists, and the ideas involved were clarified in my mind by conversations with John Moore. In the long §6 we apply our methods to a local ring R and obtain generalizations of results of Serre and Eilenberg. In particular, Theorem 8 gives the correct lower bound for the Betti numbers of a nonregular local ring. I wish to thank Zariski and Artin for several stimulating general discussions in connection with these problems.

1. R-algebras

Let R be a commutative Noetherian ring with unit element. In this note we shall use the brief term *R-algebra* to denote an associative algebra X over R in which there is defined an R-linear mapping $d: X \to X$, such that the following axioms are satisfied:

(1) X is graded, i.e. $X = \sum_{\lambda=-\infty}^{\infty} X_\lambda$ is the direct sum of R-modules X_λ such that $X_\lambda X_\mu \subset X_{\lambda+\mu}$.

(2) $X_\lambda = 0$ for $\lambda < 0$; X has a unit element $1 \in X_0$ such that $X_0 = R1$; and X_λ is a finitely generated R-module for $\lambda > 0$.

(3) X is strictly skew-commutative, that is:

$$xy = (-1)^{\lambda\mu} yx, \qquad \text{for } x \in X_\lambda,\ y \in X_\mu$$

and

$$x^2 = 0, \qquad \text{for } x \in X_\lambda,\ \lambda \text{ odd}.$$

(4) The map d is a skew derivation of degree -1, that is, $dX_\lambda \subset X_{\lambda-1}$ for all λ, $d^2 = 0$, and

$$(*) \qquad d(xy) = (dx)y + (-1)^\lambda x\,(dy), \qquad \text{for } x \in X_\lambda,\ y \in X_\mu.$$

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