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THE ANALYTIC RANK OF $J_0(q)$ AND ZEROS OF AUTOMORPHIC L-FUNCTIONS

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1. Introduction. This paper is motivated by the conjecture of Birch and Swinnerton-Dyer relating the rank of the Mordell-Weil group of an abelian variety defined over a number field with (in its crudest form) the order of vanishing of its Hasse-Weil L-function at the central critical point. Mestre [Mes] began the study of the implications of this conjecture towards providing upper bounds for the rank. He used "explicit formulae" similar to that of Riemann-Weil and assumed the analytic continuation and (perhaps more significantly) the Riemann hypothesis for those L-functions.

Brumer [Br1] first studied the special case of the Jacobian variety $J_0(q)$ of the modular curve $X_0(q)$. This is an abelian variety defined over $\mathbf{Q}$ of dimension about $q/12$. Here analytic continuation is known, by the work of Eichler and Shimura [Sh1]. Assuming only the Riemann hypothesis for the L-functions of automorphic forms (of weight 2 and level q), Brumer proved

$$\operatorname{rank}_{\mathrm{a}} J_0(q) \leqslant \left(\frac{3}{2}+o(1)\right) \dim J_0(q)$$

and conjectured that

$$\operatorname{rank} J_0(q) = \operatorname{rank}_{\mathrm{a}} J_0(q) \sim \frac{1}{2} \dim J_0(q)$$

(based on the fact that the sign of the functional equation for the automorphic L-functions of weight 2 and level q is approximately half the time $+1$ and half the time -1).

Other authors, notably Murty [Mur] (who first applied the Petersson formula in this context), considered the same problem. Most recently, Luo, Iwaniec, and Sarnak [LIS], using the same assumptions, proved an estimate

$$\operatorname{rank}_{\mathrm{a}} J_0(q) \leqslant \big(c+o(1)\big) \dim J_0(q)$$

for some (explicit) constant $c<1$. This turns out to be quite significant in light of the general conjectures of Katz and Sarnak [KaS] on the distribution of zeros of families of L-functions.

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