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# THE REDUCED BAUTIN INDEX OF PLANAR VECTOR FIELDS

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**0. Introduction.** The motivation for this paper comes from the so-called local version of the sixteenth Hilbert problem. Consider a polynomial vector field of degree $q$,

$$W_{\underline{a},\underline{b}} = x\partial_y - y\partial_x + \sum_{2\le i+j\le q} a_{ij}x^i y^j \partial_x + b_{ij}x^i y^j \partial_y,$$

the $a_{ij}$ and $b_{ij}$ being real or complex. This vector field is a deformation of the vector field $x\partial_y - y\partial_x$ whose trajectories are concentric circles around 0. We prove in this paper a precise version of the following assertion: for any compact $K$ in the space of the $(a_{ij}, b_{ij})$, there exist a number $p(q)$ and a neighborhood $U(q,K)$ of 0 such that for $(\underline{a},\underline{b}) \in K$, either 0 is again a center of $W_{\underline{a},\underline{b}}$ (i.e., 0 is an elliptic nondegenerate singular point of $W$, and $W$ is integrable near 0) or $W_{\underline{a},\underline{b}}$ has at most $p(q)$ limit cycles in $U(q,K)$. The local sixteenth Hilbert problem consists of finding explicit expressions for $U(q,K)$ and $p(q)$. This problem is solved only for $q=2$ by the so-called Bautin theorem (see [B], [Ya]). Bautin considered the Poincaré first return map around the origin restricted to a line with coordinate $X$ as a series $F_z(X)$ in $X$ with coefficients depending on the parameters $z=(a_{ij}, b_{ij})$. The limit cycles correspond to the zeroes of $F_z(X)-X$. Given a series

$$S_z(X) = \sum_{k=0}^{\infty} a_k(z)X^k$$

in one variable $X$ with polynomial coefficients $a_k(z) \in \mathbf{K}[z_1,\dots,z_n]$, $\mathbf{K}=\mathbf{R}$ or $\mathbf{C}$, Bautin then considered in [B] the ideal $I$ of $\mathbf{K}[z]$ generated by all $a_k(z)$. Since the polynomial ring is noetherian, there is a smallest integer $d$ such that $a_0,\dots,a_d$ generate $I$. This number is the Bautin index of the series $S_z(X)$. In special cases, Bautin was able to bound the number of zeroes of $S_z(X)$ and, hence, the number of limit cycles, in a function of $d$ and then to bound $d$ itself when $q=2$. More generally, when the series is an $A_0$-series in the sense of Briskin-Yomdin (see Section 2 and [BY]), for each $z$, one can bound by $d$ the number of zeroes in $X$ of the series $S_z(X)$ that lie inside a disk of radius $\mu_1(1+|z|)^{-\mu_2}$ centered at 0, where $\mu_1, \mu_2$ are positive constants depending on $S_z(X)$ (see [FY]).

In this paper, following [BY], we retain the fact that the Poincaré first return map (we call it simply the Poincaré return map in this paper) is an $A_0$-series, and we

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