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ON WEYL'S INEQUALITY, HUA'S LEMMA, AND EXPONENTIAL SUMS OVER BINARY FORMS

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1. Introduction. The remarkable success enjoyed by the Hardy-Littlewood method in its application to diagonal diophantine problems rests in large part on the theory of exponential sums in a single variable. Following almost a century of intense investigations, the latter body of knowledge has reached a mature state that, although falling short of what is expected to be true, nonetheless suffices for the majority of applications. By contrast, exponential sums in many variables remain poorly understood, and, consequently, applications of the Hardy-Littlewood method to problems concerning the solubility of systems of forms in many variables are fraught with difficulties. While the methods of Weyl and of Vinogradov have been extended to estimate exponential sums in many variables (see, in particular, Arkhipov and Karatsuba [1], Arkhipov, Karatsuba and Chubarikov [2], Tartakovsky [26], Davenport [11], [12], [13], [14], Birch [4] and Schmidt [25]), in almost all circumstances the strength of the ensuing estimates is considerably inferior to that of corresponding estimates for Weyl sums in a single variable. Moreover, one is obliged to work under various hypotheses of a geometric nature. Indeed, it is only for exponential sums over polynomials diagonalisable over $\mathbb{C}$ and for nonsingular cubic polynomials and their close kin that we have estimates of strength to match those for corresponding exponential sums of a single variable (see Chowla and Davenport [10], Birch and Davenport [5], Heath-Brown [20] and Hooley [21], [22], [23]). The purpose of this paper is to develop estimates for exponential sums over binary forms of strength comparable to the best available for corresponding exponential sums of a single variable and, moreover, without any serious geometric hypotheses on the form. Since binary forms of degree exceeding 3 in general fail to diagonalise over $\mathbb{C}$, it should be evident that our conclusions go beyond those of Birch and Davenport [5]. Our hope is that the methods described herein may spawn ideas for improved treatments of exponential sums in many variables.

Before describing our main conclusions, we require some notation. Suppose that $\Phi(x,y) \in \mathbb{Z}[x,y]$ is a binary form of degree d exceeding 1. Then we say that Φ is *degenerate* if there exist complex numbers α and β such that $\Phi(x,y)$ is identically equal to $(\alpha x+\beta y)^d$. It is easily verified that when $\Phi(x,y)$ is degenerate, then there

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