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DISKS WITH BOUNDARIES IN TOTALLY REAL AND LAGRANGIAN MANIFOLDS

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1. Introduction. An *analytic disk* in $\mathbb{C}^n$ is a map $f : D \to \mathbb{C}^n$, where D is the closed unit disk, which is holomorphic on the open disk D^o and smooth up to the boundary. One can also consider disks that are not smooth at every point of bD. An H^∞*-disk* is a bounded holomorphic map $f : D^o \to \mathbb{C}^n$. One says that the boundary of f is contained in a compact set $X \subseteq \mathbb{C}^n$ if the almost-everywhere defined boundary values $f^*(e^{i\theta})$ are contained in X for almost all θ. The maximum principle shows that $f(D^o)$ is contained in the polynomially convex hull of X. There is also an intermediate notion of a *nearly smooth analytic disk* (n.s.a.d.); this is an H^∞-disk $f : D^o \to \mathbb{C}^n$ that extends to be smooth on all of D except for (at most) a single point of bD (usually taken to be $1 \in bD$). One says that the boundary of f lies in X if the image by f of bD with the single point deleted is contained in X. The main result of [A] is that if L is an n-dimensional compact totally real manifold in $\mathbb{C}^n$, then there exists a nonconstant n.s.a.d. with boundary in L. This was proved by adapting an argument of Gromov [G] that was a cornerstone in his theory of pseudoholomorphic curves. Gromov's theorem, for the special case of $\mathbb{C}^n$, is that if L is an n-dimensional compact Lagrangian manifold in $\mathbb{C}^n$, then there exists a nonconstant analytic disk with boundary in L.

The object here is to obtain more information about these disks. Our motivation comes from the simple example of the standard torus $\mathbb{T}^3$ in $\mathbb{C}^3$, where there are analytic disks in each of the three "faces" $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = 1, |z_2| = 1\}$, $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = 1, |z_3| = 1\}$, and $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_2| = 1, |z_3| = 1\}$. Our result is that, analogously, contained in each of the faces (appropriately defined) associated with an n-dimensional totally real L in $\mathbb{C}^n$, there is a nonconstant n.s.a.d. with boundary in L. Likewise, in the Lagrangian case considered by Gromov, there exists a nonconstant analytic disk in each face. As with $\mathbb{T}^3$, the faces associated with L are related to the images of L under complex linear maps.

Let ϕ be a complex linear function on $\mathbb{C}^n$. For L compact in $\mathbb{C}^n$, we define a compact subset L_ϕ of $\mathbb{C}$ as follows: If $\mathbb{C} \setminus \phi(L)$ has only a finite number of components, then we take $L_\phi = \phi(L)$. In general, let $\mathcal{U}_\phi$ be the union of a finite family of components of $\mathbb{C} \setminus \phi(L)$, including the unbounded component and set $L_\phi = \mathbb{C} \setminus \mathcal{U}_\phi$. Thus, L_ϕ

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