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GENERATING FUNCTIONS FOR THE NUMBER OF CURVES ON ABELIAN SURFACES

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1. Introduction. Let X be an Abelian surface and let C be a holomorphic curve in X representing a primitive homology class. For n and g satisfying $C\cdot C = 2g-2+2n$, there is a g-dimensional space of curves of genus g in the class of C. To define an enumerative problem, one must impose g constraints on the curves. There are two natural ways to do this. One way is to count the number of curves passing through g generic points, which we denote $N_{g,n}(X,C)$. The second way is to count the number of curves in the fixed linear system $|C|$ passing through $g-2$ generic points, which we denote $N_{g,n}^{FLS}(X,C)$. We define (modified) Gromov-Witten invariants that compute the numbers $N_{g,n}(X,C)$ and $N_{g,n}^{FLS}(X,C)$, and we prove that they do not depend on X or C but are universal numbers henceforth denoted by $N_{g,n}$ and $N_{g,n}^{FLS}$. Our main theorem computes these numbers as the Fourier coefficients of quasi-modular forms. Note that X does not contain any genus-0 curves, so implicitly $g>0$ throughout.

THEOREM 1.1 (Main theorem). *The universal numbers $N_{g,n}^{FLS}$ and $N_{g,n}$ are given by the following generating functions:*

$$\sum_{n=0}^{\infty} N_{g,n} q^{n+g-1} = g\big(DG_2\big)^{g-1},$$

$$\sum_{n=0}^{\infty} N_{g,n}^{FLS} q^{n+g-1} = \big(DG_2\big)^{g-2} D^2 G_2$$

$$= (g-1)^{-1} D\Big(\big(DG_2\big)^{g-1}\Big),$$

where D is the operator $q(d/dq)$ and G_2 is the Eisenstein series; that is,

$$G_2(q) = -\frac{1}{24} + \sum_{k=1}^{\infty}\left(\sum_{d|k} d\right) q^k.$$

Note that the right-hand sides are quasi-modular forms (the set of quasi-modular forms is an algebra generated by modular forms, G_2, and D, cf. [5]), and the left-hand

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