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MOTIVIC EXPONENTIAL INTEGRALS AND A MOTIVIC THOM-SEBASTIANI THEOREM

JAN DENEF AND FRANÇOIS LOESER

1. Introduction

1.1. Let f and f' be germs of analytic functions on smooth complex analytic varieties X and X' and consider the function $f \oplus f'$ on $X \times X'$ given by $f \oplus f'(x, x') = f(x) + f'(x')$. The Thom-Sebastiani theorem classically states that the monodromy of $f \oplus f'$ on the nearby cycles is isomorphic to the product of the monodromy of f and the monodromy of f'. (In the original form of the theorem in [16], the functions were assumed to have isolated singularities.) It is now a common idea that the Thom-Sebastiani theorem is best understood by using Fourier transformation and exponential integrals because of the formula

$$\int \exp\big(t(f \oplus f')\big) = \int \exp(tf) \cdot \int \exp\big(tf'\big). \tag{1.1}$$

Indeed, by using asymptotic expansions of such integrals for $t \to \infty$, A. Varchenko proved a Thom-Sebastiani theorem for the Hodge spectrum in the isolated singularity case [20] (see also [14]), the general case being due to M. Saito (see [19], [11], and [13]).

The aim of the present paper is to give a motivic meaning to equation (1.1) and to deduce a motivic Thom-Sebastiani theorem. To explain our approach, we begin by reviewing some known results on p-adic exponential integrals.

1.2. Let K be a finite extension of $\mathbf{Q}_p$. Let us denote by R the valuation ring of K, by P the maximal ideal of R, and by k the residue field of K. The cardinality of k is denoted by q, so $k \simeq \mathbf{F}_q$. For z in K, $\operatorname{ord} z \in \mathbf{Z} \cup \{+\infty\}$ denotes the valuation of z, $|z| = q^{-\operatorname{ord} z}$, and $\operatorname{ac}(z) = z\pi^{-\operatorname{ord} z}$, where π is a fixed uniformizing parameter for R.

Let $f \in R[x_1, \ldots, x_m]$ be a nonconstant polynomial. Let $\Phi : R^m \to \mathbf{C}$ be a locally constant function with compact support. Let α be a character of $R^\times$, that is, a morphism $R^\times \to \mathbf{C}^\times$ with finite image. For i in $\mathbf{N}$, set

$$Z_{\Phi,f,i}(\alpha) = \int_{\{x \in R^m \mid \operatorname{ord} f(x) = i\}} \Phi(x)\alpha\big(\operatorname{ac} f(x)\big)|dx|,$$

where $|dx|$ denotes the Haar measure on K^m, normalized so that R^m is of measure 1.

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