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ADDENDUM TO "A HILBERT SPACE OF DIRICHLET SERIES AND SYSTEMS OF DILATED FUNCTIONS IN $L^2(0,1)$"

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In the present note we comment on some relevant references of which we were unaware when we published our work in the *Duke Mathematical Journal* in 1997 [HLS]. We also take the opportunity to report about the present state of a conjecture in this work. We would like to thank M. Balazard, J. Neuwirth, and E. Saksman for valuable pieces of information.

The main problem treated in our work was to find necessary and sufficient conditions for completeness and basis properties of the system $\varphi(x), \varphi(2x), \varphi(3x), \dots$ of functions in $L^2(0,\pi)$, where

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(nx), \quad \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

Thus series of the type

$$c_1\varphi(x) + c_2\varphi(2x) + c_3\varphi(3x) + \cdots, \quad \sum_{n=1}^{\infty} |c_n|^2 < \infty, \tag{1}$$

are considered. The auxiliary Dirichlet series

$$\mathcal{S}\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + it, \tag{2}$$

plays a central role. We assume that $a_1 \neq 0$. As in [HLS], we denote by $\mathcal{H}$ the Hilbert space of square-summable Dirichlet series.

In 1944 A. Wintner [W2] introduced series (2) and derived a necessary condition for the L^2-convergence of *all* series (1) with square-summable coefficients: *the auxiliary Dirichlet series (2) has an extension as an analytic and bounded function to the half-plane* $\sigma > 0$. A. Beurling in [B] used the same Dirichlet series in a seminar in 1945. In hindsight one may say that Wintner's condition means that $\sum_{n=1}^{\infty} a_n n^{-s}$ is a *multiplier*. This characterization of the multipliers is the content of our Theorem 3.1.

Wintner also gave some sufficient conditions for the system $\{\varphi(nx)\}_n$ to be complete in $L^2(0,\pi)$. In particular, he studied the example

$$\varphi(x) = \sum_{n=1}^{\infty} n^{-\lambda} \sin(nx)$$

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