Vol. 99, No. 1 DUKE MATHEMATICAL JOURNAL © 1999

A COMBINATORIAL DESCRIPTION OF THE SPECTRUM FOR THE TSETLIN LIBRARY AND ITS GENERALIZATION TO HYPERPLANE ARRANGEMENTS

PAT BIDIGARE, PHIL HANLON, AND DAN ROCKMORE

1. Introduction. Imagine a collection of books labeled 1 through n arranged in a row in some order. We reorganize the row of books by successively choosing a book at random: choosing book i with probability w_i and moving it to the front of the row. This "move-to-front rule" determines an interesting Markov chain on the set of arrangements of the books. If σ and τ denote any two orderings of the books, then the probability of transition from σ to τ is w_i if and only if τ is obtained from σ by moving book i to the front. This Markov chain is commonly called the *Tsetlin library* or *move-to-front scheme*.

Due to its use in computer science as a standard scheme for dynamic file maintenance as well as cache maintenance (cf. [Do], [FHo], and [P]), the move-to-front rule is a very well-studied Markov chain. A primary resource for this problem is Fill's comprehensive paper [F], which derives the transition probabilities for any number of steps of the chain and the eigenvalues with corresponding idempotents and discusses the rate of convergence to stationarity. Its thorough bibliography contains a wealth of pointers to the relevant literature.

Of particular interest is the spectrum of this Markov chain. In general, knowledge of the eigenvalues for the transition matrix of a Markov chain can give some indication of the rate at which the chain converges to its equilibrium distribution. In the case of the Tsetlin library, the eigenvalues have an elegant formula, discovered (independently) by P. Donnelly [Do], S. Kapoor and E. Reingold [KR], and R. Phatarfod [P].

THEOREM 1.1. *The distinct eigenvalues for the move-to-front rule are indexed by subsets* $A \subseteq \{1,\ldots,n\}$ *and given by*

$$\lambda_A = \sum_{i\in A} w_i. \tag{1}$$

The multiplicity of λ_A *is the number of derangements (permutations with no fixed points) of the set* $\{1,\ldots,n-|A|\}$

In this paper we study a sequence of generalizations of the Tsetlin library, culminating in a generalization of the setting of central hyperplane arrangements. In each

Received 17 February 1997. Revision received 1 September 1998.
1991 *Mathematics Subject Classification*. Primary 60J15; Secondary 06A07.
Hanlon partially supported by National Science Foundation grant number 9500979.
Rockmore partially supported by the National Science Foundation, PFF, and ARPA.