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QUANTUM COHOMOLOGY OF THE MODULI SPACE OF STABLE BUNDLES OVER A RIEMANN SURFACE

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1. Introduction. Let Σ be a compact Riemann surface of genus $g \geq 2$, which can be also considered as a smooth complex curve of genus g, and fix a line bundle Λ on Σ of degree 1. The central object of our study is the moduli space M_Σ of rank-2 stable vector bundles on Σ with determinant Λ, which is a smooth complex variety of complex dimension $3g-3$, and hence it is a smooth symplectic manifold of dimension $6g-6$. The symplectic deformation class of M_Σ only depends on g and not on the particular complex structure on Σ. This is seen by identifying M_Σ with the space of flat $SO(3)$-connections with nontrivial second Stiefel-Whitney class w_2 modulo the gauge transformations that can be lifted to $SU(2)$ (see [12] and [10]).

The manifold $X = M_\Sigma$ is a positive symplectic manifold with $\pi_2(X) = \mathbb{Z}$. For such a manifold X, its quantum cohomology $QH^*(X)$ is well defined (see [15], [16], [9], and [13]). As vector spaces, $QH^*(X) = H^*(X)$ (rational coefficients are understood), but the multiplicative structure is different. Let A denote the positive generator of $\pi_2(X)$, that is, the generator such that the symplectic form evaluated on A is positive. Let $N = c_1(X)[A] \in \mathbb{Z}_{>0}$. Then there is a natural $\mathbb{Z}/2N\mathbb{Z}$-grading for $QH^*(X)$, which comes from reducing the $\mathbb{Z}$-grading of $H^*(X)$. (For the case $X = M_\Sigma$, $N = 2$, so $QH^*(M_\Sigma)$ is $\mathbb{Z}/4\mathbb{Z}$-graded.) The ring structure of $QH^*(X)$, called quantum multiplication, is a deformation of the usual cup product for $H^*(X)$. For $\alpha \in H^p(X)$, $\beta \in H^q(X)$, we define the quantum product of α and β as

$$\alpha \cdot \beta = \sum_{d \geq 0} \Phi_{dA}(\alpha, \beta),$$

where $\Phi_{dA}(\alpha, \beta) \in H^{p+q-2Nd}(X)$ is given by $\langle \Phi_{dA}(\alpha, \beta), \gamma \rangle = \Psi^X_{dA}(\alpha, \beta, \gamma)$, the Gromov-Witten invariant, for all $\gamma \in H^{\dim X - p - q + 2Nd}(X)$. One has $\Phi_0(\alpha, \beta) = \alpha \cup \beta$. The other terms are the quantum correction terms, and they all live in lower degree parts of the cohomology groups. It is a fact [16] that the quantum product gives an associative and graded commutative ring structure.

To define the Gromov-Witten invariant, let J be a generic, almost complex structure, compatible with the symplectic form. Then for every 2-homology class dA, $d \in \mathbb{Z}$, there is a moduli space $\mathcal{M}_{dA}$ of pseudoholomorphic rational curves (with respect to J) $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = dA$. Note that $\mathcal{M}_0 = X$ and that $\mathcal{M}_{dA}$ is empty

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