Vol. 98, No. 3 DUKE MATHEMATICAL JOURNAL © 1999

ON QUANTUM COHOMOLOGY RINGS OF PARTIAL FLAG VARIETIES

IONUŢ CIOCAN-FONTANINE

0. Introduction. The main goal of this paper is to give a unified description for the structure of the small quantum cohomology rings for all projective homogeneous spaces $SL_n(\mathbb{C})/P$, where P is a parabolic subgroup.

The quantum cohomology ring of a smooth projective variety, or more generally of a symplectic manifold X, has been introduced by string theorists (see [Va] and [W1]). Roughly speaking, it is a deformation of the usual cohomology ring, with parameter space given by $H^*(X)$. The multiplicative structure of quantum cohomology encodes the enumerative geometry of rational curves on X. In the past few years, the highly nontrivial task of giving a rigorous mathematical treatment of quantum cohomology has been accomplished both in the realm of algebraic and symplectic geometry. In various degrees of generality, this can be found in [Beh], [BehM], [KM], [LiT1], [LiT2], [McS], and [RT], as well as in the surveys [FP] and [T].

If one restricts the parameter space to $H^{1,1}(X)$, one gets the *small quantum cohomology ring*. This ring, in the case of partial flag varieties, is the object of the present paper. In order to state our main results, we first describe briefly the "classical" side of the story.

We interpret the homogeneous space $F := SL_n(\mathbb{C})/P$ as the complex, projective variety, parametrizing flags of *quotients* of $\mathbb{C}^n$ of given ranks, say, $n_k > \cdots > n_1$.

By a classical result of Ehresmann [E], the integral cohomology of F can be described geometrically as the free abelian group generated by the *Schubert classes*. These are the (Poincaré duals of) fundamental classes of certain subvarieties $\Omega_w \subset F$, one for each element of the subset $S := S(n_1, \ldots, n_k)$ of the symmetric group S_n, consisting of permutations w with descents in $\{n_1, \ldots, n_k\}$.

A description of the multiplicative structure is provided by yet another classical theorem, due to Borel [Bor], which gives a presentation for $H^*(F, \mathbb{Z})$. Specifically, let $\sigma_1^1, \ldots, \sigma_{n_1}^1, \sigma_1^2, \ldots, \sigma_{n_2-n_1}^2, \ldots, \sigma_1^{k+1}, \ldots, \sigma_{n-n_k}^{k+1}$ be n independent variables. Define A_n to be the block-diagonal matrix $\mathrm{diag}(D_1, D_2, \ldots, D_{k+1})$, where

$$D_j := \begin{pmatrix} \sigma_1^j & \sigma_2^j & \cdots & \sigma_{n_j-n_{j-1}-1}^j & \sigma_{n_j-n_{j-1}}^j \\ -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

Received 11 December 1997. Revision received 27 May 1998.
1991 *Mathematics Subject Classification*. Primary 14M15; Secondary 14N10.
Author's work supported by a Mittag-Leffler Institute postdoctoral fellowship.