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POSITIVITY OF DUNKL'S INTERTWINING OPERATOR

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1. Introduction and results. In recent years, the theory of Dunkl operators has found a wide area of applications in mathematics and mathematical physics. Besides their use in the study of multivariable orthogonality structures associated with root systems (see, for example, [D1], [D2], [He], [vD], and [R]), these operators are closely related to certain representations of degenerate affine Hecke algebras (see [C], [O2], and, for some background, [Ki]). Moreover, they have been successfully involved in the description and solution of Calogero-Moser-Sutherland–type quantum many-body systems; among the wide literature in this context, we refer to [P], [LV] and [BF].

Let $G \subset O(N, \mathbb{R})$ be a finite reflection group on $\mathbb{R}^N$. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by σ_α the reflection in the hyperplane orthogonal to α; that is,

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{|\alpha|^2}\alpha,$$

where $\langle .,. \rangle$ denotes the Euclidean scalar product on $\mathbb{R}^N$ and $|x| := \sqrt{\langle x, x\rangle}$. We also use the notation $\langle .,. \rangle$ for the bilinear extension of the Euclidean scalar product to $\mathbb{C}^N \times \mathbb{C}^N$, while $z \mapsto |z|$ is the standard Hermitean norm on $\mathbb{C}^N$. Further, let R be the root system of G, normalized such that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$, and fix a positive subsystem R_+ of R. We recall from the general theory of reflection groups (see, e.g., [Hu]) that the set of reflections in G coincides with $\{\sigma_\alpha, \alpha \in R_+\}$ and that the orbits in R under the natural action of G correspond to the conjugacy classes of reflections in G. A function $k : R \to \mathbb{C}$ is called a multiplicity function on R if it is G-invariant. We write $\operatorname{Re} k \geq 0$ if $\operatorname{Re} k(\alpha) \geq 0$ for all $\alpha \in R$, and $k \geq 0$ if $k(\alpha) \geq 0$ for all $\alpha \in R$.

The Dunkl operators associated with G are first-order differential-difference operators on $\mathbb{R}^N$ which are parametrized by some multiplicity function k on R. For $\xi \in \mathbb{R}^N$, the corresponding Dunkl operator $T_\xi(k)$ is given by

$$T_\xi(k) f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}, \quad f \in C^1\big(\mathbb{R}^N\big).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. As k is G-invariant, the above definition is independent of the choice of R_+. In case $k = 0$, the $T_\xi(k)$ reduce to the corresponding directional derivatives. The operators $T_\xi(k)$ were introduced and

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