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SCHUBERT CALCULUS ON THE ARITHMETIC GRASSMANNIAN

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Let $\overline{G}$ be the arithmetic Grassmannian over Spec $\mathbb{Z}$ with the natural invariant Kähler metric on $\overline{G}(\mathbb{C})$. We study the combinatorics of the arithmetic Schubert calculus in the Arakelov Chow ring $CH(\overline{G})$. We obtain formulas for the arithmetic Littlewood-Richardson numbers and the Faltings height of G under the Plücker embedding, using "rim hook operations" on Young diagrams. An analysis of the duality involution leads to new combinatorial relations among Kostka numbers.

1. Introduction. Arakelov geometry provides a method of measuring the complexity of a system of diophantine equations. Such a system defines an arithmetic variety in projective space, which is then studied using techniques of intersection theory and hermitian complex geometry. The arithmetic complexity of this variety is controlled by numerical invariants called *heights*. Although their exact computation is difficult, often a good bound for these numbers is enough to prove finiteness results.

The modern theory, developed by Gillet and Soulé [GS1], attaches to each arithmetic variety X a large ring, the *arithmetic Chow ring*. Following Faltings [F], the height of X is defined as its arithmetic degree with respect to the canonical hermitian line bundle, in analogy with the geometric notion of degree. More generally, one expects that all concepts and results from geometric intersection theory should have analogues over the integers (cf. [S]).

There are very few examples where explicit formulas for heights are known; their calculation is often equivalent to evaluating intricate fiber integrals. For varieties whose complexifications are hermitian symmetric spaces, such as Grassmannians, a smaller *Arakelov Chow ring* is available, which is a subring of the larger one. In this case there are more computational tools at hand: one reduces the problem to a calculation of secondary characteristic classes called *Bott-Chern forms*. These forms are objects of pure complex geometry and are defined with no reference to arithmetic at all.

Products in the Arakelov Chow ring of projective space were computed in the foundational work of Gillet and Soulé [GS2]. A corresponding analysis for Grassmannians was done by Maillot [Ma]; he formulated an "arithmetic Schubert calculus" analogous to the classical one. The combinatorial formulas obtained in [Ma], although explicit, were quite complicated. In this article we arrive at a simpler picture.

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