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CONSTRUCTING NEW AMPLE DIVISORS OUT OF OLD ONES

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1. Introduction. The main objective of this paper is to propose a method for constructing new ample divisors on rational surfaces by gluing two given ones.

Recall that a divisor D on an algebraic variety X is ample if the corresponding line bundle $\mathcal{O}_X(D)$ is ample, and it is called *nef* (numerically effective) if there exists an ample divisor A such that $A+kD$ is ample for every $k>0$. We refer the reader to [Dem] and [Ha1] for excellent expositions on various aspects of the theory of ample and nef line bundles.

Of fundamental importance is the determination of those classes in $\mathrm{Pic}(X)$ which are ample. Although this problem has a very simple solution for smooth curves, already in dimension 2 the problem becomes much harder. Even for relatively simple surfaces, such as rational, the complete answer is not known. Several conjectures in this direction exist; however, at the present time only estimates on the *ample cone*—the cone generated by the ample classes in $\mathrm{Pic}(X)$—are known. For example, let $d, m > 0$ and consider the divisor class

$$D = \pi^* \mathcal{O}_{\mathbb{C}P^2}(d) - m \sum_{j=1}^{N} E_j$$

on the blowup $\pi : V_N \to \mathbb{C}P^2$ of $\mathbb{C}P^2$ at $N \geq 9$ generic points. Nagata conjectured in [Nag] that *D is ample if and only if $D \cdot D > 0$*, but was able to prove this only for N*'s* that are squares. In [Xu1] Xu proved that *D is ample provided that $m/d < \sqrt{N-1}/N$*. By making a more detailed analysis of the case $m=1$, Xu proved in [Xu2] that *when $d \geq 3$ the divisor class $D = \pi^*\mathcal{O}_{\mathbb{C}P^2}(d) - \sum_{j=1}^{N} E_j$ is ample if and only if $D \cdot D > 0$* (see also [Ku] for a generalization for arbitrary surfaces and [Ang] for an analogous result for $\mathbb{C}P^3$).

A closely related problem is that of computing *Seshadri constants* of ample line bundles, which measure their local positivity. The Seshadri constant $\mathcal{E}(\mathcal{L}, p)$ of the line bundle $\mathcal{L}$ at the point $p \in X$ is defined to be *the supremum of all those $\epsilon \geq 0$ for which the $\mathbb{R}$-divisor class $\pi^*\mathcal{L} - \epsilon E$ is nef on the blowup $\pi : \widetilde{X}_p \to X$ of X at the point p with exceptional divisor E.*

Seshadri constants have been much studied by Demailly [Dem]; Ein, Küchle, and Lazarsfeld [EL], [EKL], [Laz]; and Xu [Xu3]. A considerable part of these works is devoted to computations and estimates from below on the values of these constants.

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