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MULTIPLE SPIKE LAYERS IN THE SHADOW GIERER-MEINHARDT SYSTEM: EXISTENCE OF EQUILIBRIA AND THE QUASI-INVARIANT MANIFOLD

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1. Introduction

1.1. Existence of equilibria. Consider the problem

$$\begin{cases} -\Delta u+\varepsilon^{-2}\big(u-f(u)\big)=0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=0 \quad \text{on } \partial\Omega, \\ u>0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where ε is a small parameter, $\Omega\subset\mathbf{R}^d$ is a bounded domain with C^3 boundary, and $f\in C^{1,1}(\mathbf{R})$ satisfies $f\equiv 0$ in $(-\infty,0)$, $f(u)=O(u^p)$ as $u\to\infty$, and $1<p<(d+2)/(d-2)$ (if $d=2$, then $1<p<\infty$). Additionally, we assume that there exists $q>1$ such that for $u>0$ the function $f(u)/u^q$ is nondecreasing. A typical example of the nonlinearity f satisfying the above conditions is $f(u)=u^2$ for $u>0$ and $f(u)\equiv 0$ otherwise. We should point out here that our results apply for a slightly larger class of equations. We refer the reader to Section 2 for more detailed information about f.

Equation (1.1), which is of interest in itself, also plays an important role in the analysis of steady-state solutions to systems of reaction-diffusion equations arising in biology. One well-known example is the Keller-Segel system in chemotaxis (see [28] and the references therein), whose equilibria are obtained by solving (1.1). In this paper we explore the connection between (1.1) and the activator-inhibitor system, which models biological pattern formation and was proposed by Gierer and Meinhardt in [20] and [30]. This aspect of our work is described in Section 1.2.

Problem (1.1) has been studied extensively in recent years. Most of the work has been focused on establishing the existence of special solutions to (1.1), which are known as spike-layer solutions. Intuitively, a solution u^ε of (1.1) is called the spike-layer solution if $u^\varepsilon=o(1)$ as $\varepsilon\to 0$ except at an isolated point (a single spike layer) or points (multiple spike layers). At those points (peaks), $u^\varepsilon\geq\mu>0$.

Single spike layers were studied by Ni and Takagi in a series of remarkable papers

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