Vol. 97, No. 3 DUKE MATHEMATICAL JOURNAL © 1999

ANISOTROPIC FLOWS FOR CONVEX PLANE CURVES

KAI-SENG CHOU AND XI-PING ZHU

Introduction. Modeling the dynamics of melting solids or similar phenomena has been a topic of study for a long time. For sharp interfaces, the motion of the interface, that is, the boundary of the solid, is usually related to its curvature in a certain way. One of the early models was proposed by Mullins [20] for grain boundaries. This two-dimensional model is given by the equation

$$V = k, \tag{1}$$

where V and k are, respectively, the normal velocity and curvature of the interface.

Equation (1) is sometimes called the "curve shortening problem" because it is the negative L^2-gradient flow of the length of the interface. By drawing pictures, one can be easily convinced that a simple closed curve stays simple and smooth along (1) and shrinks to a point in finite time, with the limiting shape of a circle. However, a mathematical treatment of (1) turned out to be rather delicate. In fact, a rigorous study did not exist until the early 1980s, when differential geometers considered (1) as a tool in the search for simple closed geodesics on surfaces. It was also regarded as a model case for more general curvature flows, which are believed to be important in the topological classification of low-dimensional manifolds. As a first attempt, Gage [10] proved that the isoperimetric ratio decreases along convex curves. Then in Gage and Hamilton [12], it was shown that a convex curve stays convex and shrinks to a point in finite time. Moreover, if one normalizes the flow by dilating it so that the enclosing area is constant, the normalized flow converges smoothly to a circle. Finally, Grayson [14] completed this line of investigation by showing that a simple curve evolves into a convex one before shrinks to a point. For an alternative approach to this result, one may consult Hamilton [18].

Recently Mullins's theory was generalized by Gurtin [15], [16] and by Angenent and Gurtin [4], [5] (see also the monograph of Gurtin [17]) to include anisotropy and the possibility of a difference in bulk energies between phases. Anisotropy is indispensable in dealing with crystalline materials. For perfect conductors, the temperatures in both phases are constant. The equation becomes

$$\beta(\theta)V = g(\theta)k + F, \tag{2}$$

Received 23 April 1996.

1991 *Mathematics Subject Classification*. Primary 35K57; Secondary 58E15.

Authors' research supported by an Earmarked Grant of Hong Kong.

Zhu partially supported by the Foundation for Outstanding Young Scholars and the National Science Foundation of China.