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VALUES AT INTEGERS OF HOMOGENEOUS POLYNOMIALS

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1. Introduction. Berry and Tabor conjecture in [1] that the distribution of local spacings between the eigenvalues of the quantization of the Hamiltonian for generic, completely integrable systems follow those of random numbers. (Such a distribution is often called "Poissonian.") In [8], Sarnak considers the specific case of the geodesic flow on a flat two-dimensional torus, for which the Berry-Tabor conjecture reduces to a conjecture about the spacings between the values at nonnegative integers of binary quadratic forms. In this article we study the spacings of the values at nonnegative integers of homogeneous forms of higher degree.

Consider a homogeneous polynomial $P(x_1,\dots,x_k)$ of degree k, and let B be a compact set in $\mathbf{R}^k$. Given T large, the set $\{P(\vec{x})|\vec{x}\in\mathbf{Z}^k\cap TB\}$ consists of (up to constants) T^k elements contained within a range of size T^k. Their average spacing is thus constant, so we can consider the distribution of the pair spacings $P(\vec{x}_i)-P(\vec{x}_j)$. In addition, if we order the vectors so that $P(\vec{x}_0)\le P(\vec{x}_1)\le\cdots$, we may also investigate the distribution of the consecutive spacings $P(\vec{x}_i)-P(\vec{x}_{i+1})$. We define

$$R_P(a,b,B,T)=\frac{|\{\vec{x}_i,\vec{x}_j\in(\mathbf{Z}^k\cap TB)|a\le P(\vec{x}_i)-P(\vec{x}_j)\le b\}|}{T^k} \tag{1}$$

to be the pair correlation of P, and

$$E_P(a,b,B,T)=\frac{|\{\vec{x}_i\in(\mathbf{Z}^k\cap TB)|a\le P(\vec{x}_{i+1})-P(\vec{x}_i)\le b\}|}{T^k} \tag{2}$$

to be its consecutive spacing distribution. If the λ's are distributed evenly by a random process, then with probability one

$$\lim_{T\to\infty}R_P(a,b,B,T)=c_P^2(b-a)$$

and

$$\lim_{T\to\infty}E_P(a,b,B,T)=c_P\int_a^b e^{-c_Pt}\,dt,$$

where c_P is the inverse of the average distance between consecutive values of P (that is, the density of the values of P).

A simple case when these limits cannot possibly hold is when k is even and B contains points that are negatives of each other, since then $P(\vec{x})=P(-\vec{x})$ implies that

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