Vol. 97, No. 1 DUKE MATHEMATICAL JOURNAL

LIMIT THEOREMS FOR THETA SUMS

JENS MARKLOF

Contents

1. Introduction 127
2. Basic definitions 129
3. Theta functions as functions on $\Delta_1(4)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ 131
4. Geodesic flows and horocycle flows 138
5. The distribution of values in the complex plane 143
6. The asymptotic behaviour of moments 145
7. The classical theta sum 148

1. Introduction. The classical theta sum is defined by

$$S_N(x) = N^{-1/2} \sum_{n=1}^{N} e\big(n^2 x\big), \tag{1}$$

where $e(t) = \exp(2\pi \mathrm{i} t)$. We are interested in the asymptotic behaviour of $S_N(x)$ as N tends to infinity, for arbitrary values of x in $\mathbb{R}$. The case when x is rational is the easiest. It is not hard to see that here $S_N(x) = A(x)N^{1/2} + O(N^{-1/2})$ with some constant $A(x)$ (which can be zero for certain x), for $S_N(x)$ reduces to ordinary Gauss sums. The more difficult part of giving estimates for generic values of x was first discussed by Hardy and Littlewood [3], [4], using diophantine approximation. Their methods were later refined in a number of publications, some of which we will mention here. In contrast to these approaches, we investigate the asymptotic behaviour of theta sums by means of ergodic theory, exploiting a connection to the geodesic flow and the horocycle flow on the unit tangent bundle of a certain hyperbolic surface.

In order to simplify the presentation of the main ideas, we replace the sharp cutoff in the sum (1) by a smooth one; that is, we take a C^∞-function f (with compact support, say) and consider the sum

$$\widetilde{S}_N(x) = N^{-1/2} \sum_{n\in\mathbb{Z}} f(n/N)e\big(n^2 x\big). \tag{2}$$

Choosing f to be the characteristic function of the interval $(0,1]$ would clearly lead

Received 4 September 1996. Revision received 31 October 1997.

1991 *Mathematics Subject Classification*. Primary 11F27; Secondary 11K60, 11L15, 58F17.

Author's research supported by Gottlieb Daimler- und Karl Benz-Stiftung under grant number 2.94.36.