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COUNTING RATIONAL CURVES ON K3 SURFACES

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Introduction. The aim of this paper is to explain the remarkable formula found by Yau and Zaslow [YZ] to express the number of rational curves on a K3 surface. Projective K3 surfaces fall into countably many families $(\mathcal{F}_g)_{g\geq 1}$; a surface in $\mathcal{F}_g$ admits a g-dimensional linear system of curves of genus g. A naïve count of constants suggests that such a system will contain a positive number, say, $n(g)$, of rational (highly singular) curves. The formula is

$$\sum_{g\geq 0} n(g)q^g = \frac{q}{\Delta(q)},$$

where $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the well-known modular form of weight 12, and by convention we put $n(0)=1$.

To explain the idea in a nutshell, take the case $g=1$. We thus look at K3 surfaces with an elliptic fibration $f: S \to \mathbf{P}^1$, and we ask for the number of singular fibres. The (topological) Euler-Poincaré characteristic of a fibre C_t is zero if C_t is smooth, is 1 if it is a rational curve with one node, is 2 if it has a cusp, and so on. From the standard properties of the Euler-Poincaré characteristic, we get $e(S) = \sum_t e(C_t)$; hence, $n(1) = e(S) = 24$, and this number counts nodal rational curves with multiplicity 1, cuspidal rational curves with multiplicity 2, and so on.

The idea of Yau and Zaslow is to generalize this approach to any genus. Let S be a K3 surface with a g-dimensional linear system Π of curves of genus g. The role of f is played by the morphism $\bar{\mathcal{J}}\mathcal{C} \to \Pi$, whose fibre over a point $t \in \Pi$ is the compactified Jacobian $\bar{J}C_t$. To apply the same method, we would like to prove the following facts.

(1) The Euler-Poincaré characteristic $e(\bar{\mathcal{J}}\mathcal{C})$ is the coefficient of q^g in the Taylor expansion of $q/\Delta(q)$.

(2) $e(\bar{J}C_t) = 0$ if C_t is not rational.

(3) $e(\bar{J}C_t) = 1$ if C_t is a rational curve with nodes as only singularities. Moreover $e(\bar{J}C_t)$ is positive when C_t is rational, and can be computed in terms of the singularities of C_t.

(4) For a generic K3 surface S in $\mathcal{F}_g$, all rational curves in Π are nodal.

The first statement is proved in Section 1, by comparing $e(\bar{\mathcal{J}}\mathcal{C})$ with the Euler-Poincaré characteristic of the Hilbert scheme $S^{[g]}$, which has been computed by

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