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AN EXACT MASS FORMULA FOR ORTHOGONAL GROUPS

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Introduction. Our mass formula concerns the orthogonal group G^φ of a nondegenerate symmetric bilinear form $\varphi : V \times V \to F$, where V is an n-dimensional vector space over an algebraic number field F. We take F to be totally real, and we assume in the introduction that φ is totally definite, though we shall also treat the indefinite case. Given a lattice L in V, we consider its genus Λ, which consists of all the lattices in V that are equivalent to L with respect to the localizations of G^φ at all nonarchimedean primes. It is well known that there exists a finite set of lattices $\{L_i\}_{i=1}^h$ such that Λ is the disjoint union $\bigsqcup_{i=1}^h \{L_i\alpha \,|\, \alpha \in G^\varphi\}$. Here, as well as in the text, we let G^φ act on V on the right. Then we define the *mass of the genus* Λ to be the sum

$$\mathfrak{m}(\varphi, \Lambda) = \sum_{i=1}^{h} [\Gamma_i : 1]^{-1}, \quad \Gamma_i = \{\alpha \in G^\varphi \,|\, L_i\alpha = L_i\}.$$

Following a pioneer work of Minkowski, Siegel showed that the mass is an infinite product $\prod_v \{2/e_v(\varphi)\}$ with certain representation densities $e_v(\varphi)$ defined at all archimedean and nonarchimedean primes v of F. He calculated these factors explicitly except for finitely many nonarchimedean primes, finding that the product consists essentially of several special values of the Dedekind zeta function and an L-function of F, but he did not give an exact form. In fact he said something to the effect that his formula was the most practicable way of expressing the relationship between the global theory of quadratic forms and the local theory, since the determination of the bad factors was a tiresome and complicated task, which Minkowski had undertaken unsuccessfully some fifty years earlier. The formulation of this problem in terms of the Tamagawa number provided a perspective applicable to a wide class of algebraic groups, but it may fairly be said that it has encouraged the researchers in this field to continue avoiding the issue.

In order to face the issue squarely, we first notice that the expression of the class number of an algebraic number field in terms of the residue of the Dedekind zeta function is given only for the maximal order, and the class number of an arbitrary order must be given relative to that of the maximal order. Therefore, in the case of orthogonal groups, it is natural to expect a clear-cut formula only for a special type of lattice. Now Eichler introduced the notion of a *maximal lattice*, which is maximal among the lattices on which the quadratic form $x \mapsto \varphi(x, x)$ takes integral values. It is with this type of lattice that we shall give our exact mass formula. Eichler himself

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