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UNIVERSAL SCHUBERT POLYNOMIALS

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1. Introduction. The aim of this paper is to introduce some polynomials that specialize to all previously known Schubert polynomials: the classical Schubert polynomials of Lascoux and Schützenberger [LS1], [M], the quantum Schubert polynomials of Fomin, Gelfand, and Postnikov [FGP], and the quantum Schubert polynomials for partial flag varieties of Ciocan-Fontanine [CF2]. There are also double versions of these universal Schubert polynomials that generalize the previously known double Schubert polynomials [L], [M], [KM], [CFF]. They describe degeneracy loci of maps of vector bundles, but in a more general setting than the previously known setting of [F2].

These universal Schubert polynomials possess many but not all algebraic properties of their classical specializations. Their extra structure makes them useful for studying their specializations, as it can be easier to find patterns before variables are specialized.

The main geometric setting to which these polynomials apply is the following. We have maps of vector bundles

$$F_1 \to F_2 \to \cdots \to F_n \to E_n \to \cdots \to E_2 \to E_1 \tag{1}$$

on a variety or scheme X, where each F_i and E_i has rank i. We do *not* assume here that the maps $F_i \to F_{i+1}$ are injective or that the maps $E_{i+1} \to E_i$ are surjective, as was the case studied in [F2]. For each w in the symmetric group S_{n+1}, there is a degeneracy locus

$$\Omega_w = \{x \in X \mid \operatorname{rank}(F_q(x) \to E_p(x)) \leqslant r_w(p,q) \text{ for all } 1 \leqslant p, q \leqslant n\}, \tag{2}$$

where $r_w(p,q)$ is the number of $i \leqslant p$ such that $w(i) \leqslant q$. Such degeneracy loci are described by the double form $\mathfrak{S}_w(c,d)$ of universal Schubert polynomials, evaluated at the Chern classes of all the bundles involved. Unlike the situation studied in [F2], where these Chern classes were determined by their first Chern classes, in the present general setting one must have more general polynomials to describe such loci. There are similar formulas when some of the bundles in (1) are missing.

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