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ON THE η-INVARIANT OF CERTAIN NONLOCAL BOUNDARY VALUE PROBLEMS

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CONTENTS

1. Introduction. A very intriguing feature of elliptic operators on compact manifolds without boundary is the locality of their indices. Specifically, if M denotes a compact Riemannian spin manifold, $S \longrightarrow M$ a spinor bundle, $E \longrightarrow M$ a hermitian coefficient bundle with unitary connection, and D^E the Dirac operator on M with coefficients in E, then, by the Atiyah-Singer theorem,

$$\operatorname{ind} D_+^E = \int_M \hat{A}(M) \wedge \operatorname{ch} E. \tag{1.1}$$

Here D_+^E arises from splitting $S \otimes E$ under the involution induced by the complex volume element on M.

If M decomposes along a compact hypersurface N as $M = M_1 \cup M_2$, with $\partial M_i = N$ for $i = 1, 2$, then one is lead to ask whether the obvious decomposition of the right-hand side in (1.1) corresponds to a decomposition of the (essentially) selfadjoint operator D^E into selfadjoint operators D_i^E, defined in M_i by suitable boundary conditions on N, such that

$$\operatorname{ind} D_{1,+}^E + \operatorname{ind} D_{2,+}^E = \operatorname{ind} D_+^E. \tag{1.2}$$

This question was answered in the affirmative by Atiyah, Patodi, and Singer [APS] who formulated the correct boundary conditions (cf. Sec. 2 for details). More importantly, the resulting index formula (2.7) displayed a new spectral invariant of selfadjoint elliptic operators (defined on N), which they called the η-invariant. It is not locally computable by a formula as in (1.1), as can be seen from its behaviour under coverings. Nevertheless, one can ask how the η-

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